

ELASTIC GRAPHS

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ABSTRACT. An *elastic graph* is a graph with an elasticity associated to each edge. It may be viewed as a network made out of ideal rubber bands. If the rubber bands are stretched on a target space there is an *elastic energy*. We characterize when a homotopy class of maps from one elastic graph to another is *loosening*, i.e., decreases this elastic energy for all possible targets. This fits into a more general framework of energies for maps between graphs.

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1. INTRODUCTION

Question 1.1. When is one network of rubber bands looser than another?

To make Question 1.1 precise, we give some definitions.

Definition 1.2. In this paper, a *graph* is a topologist's graph, a finite 1-dimensional CW complex, with multiple edges and self-loops allowed. *Maps* between graphs are continuous maps between the underlying topological spaces. In particular, they need not send vertices to vertices. For convenience, we will work with maps that are PL with respect to a (fixed) linear structure on each edge.

A *marked graph* is a pair (Γ, M) , where Γ is a graph and $M \subset \text{Vert}(\Gamma)$ is a finite subset of marked points (possibly empty). A map $f: (\Gamma_1, M_1) \rightarrow (\Gamma_2, M_2)$ between marked graphs is required to send marked points to marked points (i.e., $f(M_1) \subset M_2$). Homotopy is considered within the space of such maps. In particular, within a homotopy class the restriction of f to a map from M_1 to M_2 is fixed.

Definition 1.3. A *length graph* $K = (\Gamma, \ell)$ is a graph in which each edge e has a positive length $\ell(e)$. This gives a metric on Γ respecting the linear structure on the edges. If $f:$

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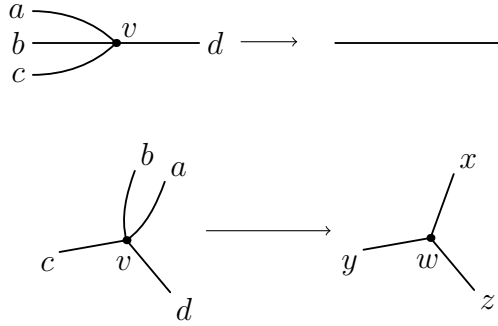


FIGURE 1. Local schematic pictures of the equilibrium condition for harmonic maps. Top: A vertex maps to an edge. Bottom: A vertex maps to a vertex.

$K_1 \rightarrow K_2$ is a PL map between length graphs, then $|f'|: K_1 \rightarrow \mathbb{R}_{\geq 0}$ is the absolute value of the derivative with respect to the metrics. It is locally constant with jump discontinuities.

Definition 1.4. A *elastic graph* $G = (\Gamma, \alpha)$ is a graph Γ in which each edge e has a positive elastic constant $\alpha(e)$. For G an elastic graph, K a length graph, and $f: G \rightarrow K$ a PL map, the *Dirichlet energy* of f is

$$(1.5) \quad \text{Dir}(f) := \int_{x \in \Gamma} |f'(x)|^2 dx.$$

(To take the derivative, we view $\alpha(e)$ as the length of the edge.) The Dirichlet energy of a (marked) homotopy class $[f]$ is defined to be

$$(1.6) \quad \text{Dir}[f] := \inf_{g \in [f]} \text{Dir}(g).$$

It is easy to see that a minimizer g for $\text{Dir}[f]$ must be *constant-derivative*: $|g'|$ is constant on each edge of g (Definition 4.5). If g is constant-derivative, we then have

$$(1.7) \quad \text{Dir}(g) = \sum_{e \in \text{Edge}(\Gamma)} \frac{\ell(g(e))^2}{\alpha(e)},$$

where $\ell(g(e))$ is the length of the image of e in a natural sense (Definition 2.2). This is the Hooke's law energy of g , where each edge is an ideal spring with resting length 0 and spring constant given by α . Physically, you could think about stretching a rubber band graph shaped like G over a system of pipes shaped like K .

Harmonic maps are maps that locally minimize the energy (1.5) or (1.7) within their homotopy class. Intuitively, a map is harmonic if it is constant-derivative and each vertex is at equilibrium, in the sense that the force pulling in one direction is less than or equal to the total force pulling it in other directions. Physically, the force pulling a vertex in the direction of an incident edge e is given by the derivative of Equation (1.7) with respect to $\ell(g(e))$, i.e., $2\ell(g(e))/\alpha(e) = 2|f'(e)|$. We will drop the irrelevant factor of 2 and refer to $|f'(e)|$ as the *tension* in the edge.

For instance, if a vertex v of G maps to the middle of an edge of K with three edges on the left and one edge on the right as on top of Figure 1, the net force to the left must equal the net force to the right:

$$|f'(a)| + |f'(b)| + |f'(c)| = |f'(d)|.$$

On the other hand, if a vertex v of G maps to a vertex w of K as on the bottom of Figure 1, we no longer have an equality. Instead, the net force pulling v into any one of the three edges incident to w can't be greater than the total force pulling it into the other two edges. This gives three triangle inequalities:

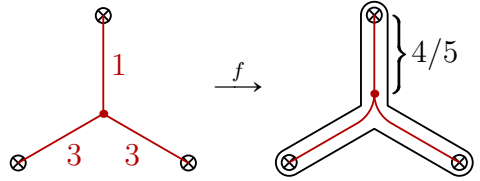
$$\begin{aligned} |f'(a)| + |f'(b)| &\leq |f'(c)| + |f'(d)| \\ |f'(c)| &\leq |f'(a)| + |f'(b)| + |f'(d)| \\ |f'(d)| &\leq |f'(a)| + |f'(b)| + |f'(c)|. \end{aligned}$$

In any homotopy class of maps to a target length graph there is a harmonic map (Theorem 5).

Example 1.8. Consider an elastic graph G and length graph K , both tripods with their ends marked, with elastic weights and lengths given by



where x varies. If $x = 3$, the map f minimizing Dirichlet energy takes the vertex of G to a point $1/5$ of the way along an edge:



The net force pulling the vertex of G upwards is $4/5$, the same as the net force pulling it downwards. The total Dirichlet energy from Equation (1.7) is

$$\text{Dir}(f) = \text{Dir}[f] = (4/5)^2 + (6/5)^2/3 + (6/5)^2/3 = 8/5.$$

On the other hand, if $0 < x \leq 2$, then the harmonic representative has the central vertex of G mapping to the central vertex of K .

We can think of Dirichlet energy as a function of the target lengths (or more precisely the target lengths and the homotopy class). We next compare these functions.

Definition 1.9. Given a homotopy class $[\phi]: G_1 \rightarrow G_2$ of maps between marked elastic graphs, we say that $[\phi]$ is *weakly loosening* if, for all marked length graphs K and marked maps $f: G_2 \rightarrow K$,

$$\text{Dir}[f \circ \phi] \leq \text{Dir}[f].$$

If the inequality is always strict, we say that $[\phi]$ is *strictly loosening*. For a finer invariant, define the *Dirichlet stretch factor* to be

$$(1.10) \quad \text{SF}_{\text{Dir}}[\phi] := \sup_{[f]: G_2 \rightarrow K} \frac{\text{Dir}[f \circ \phi]}{\text{Dir}[f]},$$

where the supremum runs over all marked length graphs K and all marked homotopy classes $[f]$, so that $[\phi]$ is weakly loosening iff $\text{SF}_{\text{Dir}}[\phi] \leq 1$. It is less obvious, but also true that $[\phi]$ is strictly loosening iff $\text{SF}_{\text{Dir}}[\phi] < 1$.

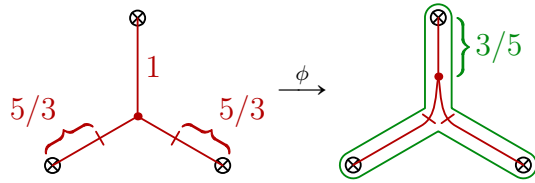
Definition 1.11. For $\phi: G_1 \rightarrow G_2$ a PL map between marked elastic graphs, the *embedding energy* of ϕ is

$$(1.12) \quad \text{Emb}(\phi) := \text{ess sup}_{y \in G_2} \sum_{x \in \phi^{-1}(y)} |\phi'(x)|.$$

We are taking the essential supremum over points $y \in G_2$, ignoring in particular vertices of G_2 , images of vertices of G_1 , and points where $\phi^{-1}(y)$ is infinite, all of which have measure zero. For a homotopy class $[\phi]$, define

$$(1.13) \quad \text{Emb}[\phi] := \inf_{\psi \in [\phi]} \text{Emb}(\psi).$$

Example 1.14. If G_1 and G_2 are both tripods with marked ends, with elastic weights $(1, 3, 3)$ and $(1, 1, 1)$, then the minimizer for $\text{Emb}(\phi)$ is not the map from Example 1.8, but instead sends the vertex of G_1 to a point $2/5$ of the way along an edge of G_2 , with $\text{Emb}(\phi) = 3/5$:



Our answer to Question 1.1 says that $\text{Emb}[\phi] = \text{SF}_{\text{Dir}}[\phi]$, and in particular the homotopy class $[\phi]$ is weakly loosening exactly when $\text{Emb}[\phi] \leq 1$. Before stating the theorem, we give another characterization of loosening, in terms of maps to an elastic graph G (rather than maps from G).

Definition 1.15. A *marked one-manifold* is a (not necessarily connected) one-manifold C with boundary, where the set of marked points is equal to ∂C . A *marked multi-curve* on a marked graph Γ is a marked one-manifold C and a marked map $c: C \rightarrow \Gamma$. (Thus, it is a union of loops on Γ and arcs between marked points of Γ .) A marked multi-curve is *strictly reduced* if it is PL and has no backtracking: on each component of C , c is either constant or has a perturbation that is locally injective. In each homotopy class of multi-curves there is an essentially unique reduced representative. If (C, c) is a marked multi-curve on Γ , then for $y \in \Gamma$, define $n_c(y)$ to be the size of $c^{-1}(y)$. If c is reduced and e is an edge of Γ , then $n_c(y)$ is constant almost everywhere on e , so we may write $n_c(e)$.

A *marked weighted multi-curve* is a marked multi-curve in which each component C_i has a positive weight w_i . For weighted multi-curves, $n_c(y)$ is the weighted count of $c^{-1}(y)$.

Definition 1.16. The *extremal length* of a reduced multi-curve c on an elastic graph $G = (\Gamma, \alpha)$ is

$$\text{EL}[c] = \text{EL}(c) := \sum_{e \in \text{Edge}(G)} \alpha(e) n_c(e)^2.$$

The extremal length of a homotopy class of marked multi-curves is the extremal length of any reduced representative.

Definition 1.17. For $[\phi]: G_1 \rightarrow G_2$ a homotopy class of maps between elastic marked graphs, the *extremal length stretch factor* is

$$(1.18) \quad \text{SF}_{\text{EL}}[\phi] := \sup_{c: C \rightarrow G_1} \frac{\text{EL}[\phi \circ c]}{\text{EL}[c]}$$

where the supremum runs over all multi-curves (C, c) on G_1 .

We are now ready to state the main result of this paper.

Theorem 1. For $[\phi]: G_1 \rightarrow G_2$ a homotopy class of maps between marked elastic graphs,

$$(1.19) \quad \text{Emb}[\phi] = \text{SF}_{\text{Dir}}[\phi] = \text{SF}_{\text{EL}}[\phi].$$

Furthermore, there is

- a PL map $\psi \in [\phi]$ realizing $\text{Emb}[\phi]$;
- a marked length graph K and a PL map $f: G_2 \rightarrow K$ realizing SF_{Dir} , in the sense that

$$\text{SF}_{\text{Dir}}[\phi] = \frac{\text{Dir}[f \circ \phi]}{\text{Dir}[f]} = \frac{\text{Dir}(f \circ \psi)}{\text{Dir}(f)};$$

and

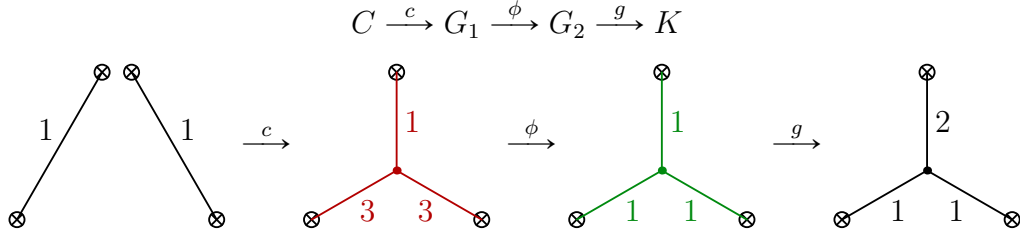
- a marked weighted multi-curve (C, c) on G_1 realizing SF_{EL} , in the sense that c and $\psi \circ c$ are both reduced and

$$\text{SF}_{\text{EL}}[\phi] = \frac{\text{EL}(\psi \circ c)}{\text{EL}(c)}.$$

Note that $\text{Emb}[\phi]$ is defined as an infimum (over the homotopy class), while $\text{SF}_{\text{Dir}}[\phi]$ and $\text{SF}_{\text{EL}}[\phi]$ are defined as suprema (over all possible targets or multi-curves). As such, Equation (1.19) helps us compute these quantities, by sandwiching the target value.

In the course of the proof in Section 6, we also give an algorithm that produces, simultaneously, the map $\psi \in [\phi]$ minimizing Emb , the pair (K, f) maximizing the ratio of Dirichlet energies, and the reduced multi-curve (C, c) maximizing the ratio of extremal lengths.

Example 1.20. Example 1.14 fits into the following sequence of maps realizing SF_{EL} and SF_{Dir} :



Here c is the evident curve map, ϕ is the map from Example 1.14, and g is the map that takes the vertex of G_2 to the vertex of K . Specifically, we have

$$\frac{\text{EL}[\phi \circ c]}{\text{EL}[c]} = \frac{4 + 1 + 1}{4 + 3 + 3} = \text{Emb}(\phi) = 3/5 = \frac{\text{Dir}[g \circ \phi]}{\text{Dir}[g]} = \frac{36/25 + 27/25 + 27/25}{4 + 1 + 1}.$$

We can furthermore generalize the target spaces in Theorem 1 considerably.

Definition 1.21. Let G be a marked elastic graph and let X be a marked length space (a length space with a distinguished finite set of points). Let $f: G \rightarrow X$ be a Lipschitz map. Define the Dirichlet energy of f by

$$\text{Dir}(f) := \int_{x \in G} |f'(x)|^2 dx$$

where $|f'(x)|$ is the best local Lipschitz constant of f in a neighborhood of x . For a homotopy class of marked maps, define

$$\text{Dir}[f] := \inf_{g \in [f]} \text{Dir}(g).$$

In this generality, we have no guarantee that minimizers for $\text{Dir}[\phi]$ exist. (See [KS93, EF01] for some cases where minimizers do exist.)

Theorem 2. For $[\phi]: G_1 \rightarrow G_2$ a homotopy class of maps between marked elastic graphs,

$$\text{Emb}[\phi] = \sup_{[f]: G_2 \rightarrow X} \frac{\text{Dir}[f \circ \phi]}{\text{Dir}[f]},$$

where the supremum runs over all marked length spaces X and all homotopy classes of marked maps $f: G_2 \rightarrow X$ with $\text{Dir}[f] > 0$.

We next put Theorem 1 in a broader context of graphical energies. There is a family of energies of maps between weighted graphs, elastic graphs, and length graphs:

$$(1.22) \quad \begin{array}{ccccc} & & \ell & & \\ & \nearrow & & \searrow & \\ \text{Weighted} & \xrightarrow{\sqrt{\text{EL}}} & \text{Elastic} & \xrightarrow{\sqrt{\text{Dir}}} & \text{Length} \\ \text{graph } W & & \text{graph } G & & \text{graph } K \\ \uparrow & & \uparrow & & \uparrow \\ \text{WR} & & \sqrt{\text{Emb}} & & \text{Lip} \end{array}$$

(Weighted graphs, Definition 2.3, are a mild generalization of weighted multi-curves.) The label on an arrow gives the appropriate energy of a map between the given types of graphs. $\text{EL}(f)$, $\text{Emb}(f)$ and $\text{Dir}(f)$ are as defined above, except that we take the square root for reasons to be explained shortly, and EL has been extended from multi-curves to maps from general weighted graphs in a natural way (Equation (2.5)). We make some additional definitions.

- For $f: W \rightarrow K$ a map from a weighted graph to a length graph, $\ell(f)$ is the weighted length of the image of f :

$$(1.23) \quad \ell(f) := \int_{x \in W} w(x) |f'(x)| dx.$$

- For $f: K_1 \rightarrow K_2$ a PL map between length graphs, $\text{Lip}(f)$ is the best global Lipschitz constant for f :

$$(1.24) \quad \text{Lip}(f) := \text{ess sup}_{x \in K_1} |f'(x)|.$$

- For $f: W_1 \rightarrow W_2$ a PL map between weighted graphs, the *weight ratio* is the maximum ratio of weights:

$$(1.25) \quad \text{WR}(c) := \text{ess sup}_{y \in W_2} \frac{\sum_{x \in f^{-1}(y)} w(x)}{w(y)}.$$

A key fact is that these energies are *submultiplicative*, in the sense that the energy of a composition of two maps is less than or equal to the product of the energies of the two pieces. To state this uniformly, we make some further definitions.

Definition 1.26. For $p \in \{1, 2, \infty\}$, a p -conformal graph G^p is

- for $p = 1$, a weighted graph;
- for $p = 2$, an elastic graph; and
- for $p = \infty$, a length graph.

Let $p, q \in \{1, 2, \infty\}$ with $p \leq q$. If we have a p -conformal graph G_1 , a q -conformal graph G_2 , and a PL map $f: G_1^p \rightarrow G_2^q$ between them, define $E_q^p(f)$ by

$$\begin{aligned} E_1^1(f) &:= \text{WR}(f) & E_2^1(f) &:= \sqrt{\text{EL}(f)} & E_\infty^1(f) &:= \ell(f) \\ E_2^2(f) &:= \sqrt{\text{Emb}(f)} & E_\infty^2(f) &:= \sqrt{\text{Dir}(f)} \\ E_\infty^\infty(f) &:= \text{Lip}(f). \end{aligned}$$

In each case, for a homotopy class $[f]$ of maps, define

$$E_q^p[f] := \inf_{g \in [f]} E_q^p(g).$$

Let $p, q, r \in \{1, 2, \infty\}$ with $p \leq q \leq r$. Suppose we have a sequence of maps

$$G_1^p \xrightarrow{f} G_2^q \xrightarrow{g} G_3^r$$

between marked graphs of the respective conformal type. Then

$$(1.27) \quad E_r^p(g \circ f) \leq E_q^p(f) E_r^q(g)$$

$$(1.28) \quad E_r^p[g \circ f] \leq E_q^p[f] E_r^q[g].$$

See Proposition 2.15 for a more detailed statement, spelling out the cases.

More generally, p -conformal graphs and energies E_q^p can be defined uniformly for real numbers p, q with $1 \leq p \leq q \leq \infty$. See Appendix A.

Theorem 1 can be interpreted as saying that some cases of Equation (1.28) are tight, as follows.

Definition 1.29. Let $p, q \in \{1, 2, \infty\}$ with $p \leq q$. If $f: G_1^p \rightarrow G_2^q$ is a PL map between marked graphs of the respective conformal type and

$$E_q^p(f) = E_q^p[f],$$

we say that f is an *energy minimizer* (in its homotopy class).

Let $p, q, r \in \{1, 2, \infty\}$ with $p \leq q \leq r$. If we have a sequence of maps

$$G_1^p \xrightarrow{f} G_2^q \xrightarrow{g} G_3^r$$

of PL maps between graphs of the respective conformal type, we say that this sequence is *tight* if $g \circ f$ is an energy minimizer and Equation (1.27) is sharp:

$$E_r^p[g \circ f] = E_r^p(g \circ f) = E_q^p(f) E_r^q(g).$$

We may similarly say that a longer sequence of maps is tight.

Lemma 1.30. *Let $G_1^p \xrightarrow{f} G_2^q \xrightarrow{g} G_3^r$ be a sequence of maps. If the sequence is tight, then f and g are also energy minimizers. Conversely, if f and g are energy minimizers and $E_r^p[g \circ f] = E_q^p[f] E_r^q[g]$, then the sequence is tight.*

Proof. In a general sequence of maps, we have inequalities:

$$\begin{aligned} E_r^p[g \circ f] &\leq E_r^p(g \circ f) \leq E_q^p(f) E_r^q(g) \\ E_r^p[g \circ f] &\leq E_q^p[f] E_r^q[g] \leq E_q^p(f) E_r^q(g). \end{aligned}$$

The two conditions in the lemma assert that the inequalities in the top line or in the bottom line are equalities. In either case, the inequalities in the other line must be equalities as well. \square

In this language, Theorem 1 says that, for every homotopy class of maps $[\phi]: G_1 \rightarrow G_2$ between marked elastic graphs, there is a corresponding tight sequence

$$C \xrightarrow{c} G_1 \xrightarrow{\psi} G_2 \xrightarrow{f} K$$

with $\psi \in [\phi]$, (C, c) a weighted multi-curve on G_1 , and K a length graph. See Proposition 6.12 for more details about the structure of maps that minimize $\text{Emb}(\psi)$.

In the course of the paper, we also prove several other tightness results.

- In Section 3, we prove Theorem 3, which gives a version of the max-flow/min-cut theorem in the context of maps between graphs. This gives a characterization of which maps minimize E_p^1 (for any p).
- In Section 4, we prove Theorem 4, which shows that for any homotopy class $[\phi]: K_1 \rightarrow K_2$ of maps between marked length graphs, there is a tight sequence

$$C \xrightarrow{c} K_1 \xrightarrow{\psi} K_2$$

where $\psi \in [\phi]$. This is a theorem of White, used by Bestvina [Bes11]: the minimal Lipschitz stretch factor between metric graphs equals the maximal ratio by which lengths of multi-curves are changed. We also extend the graphs we are looking at to allow *weak graphs*, where lengths are allowed to be zero, and prove the corresponding tightness result (Theorem 4').

- In Section 5, we prove Theorem 5, which shows that for any homotopy class $[f]: G \rightarrow K$ of maps from a marked elastic graph to a marked length graph, there is a tight sequence

$$C \xrightarrow{c} G \xrightarrow{g} K$$

where $g \in [f]$. Theorem 5 also characterizes the energy minimizers for $\text{Dir}(g)$ (harmonic maps). Again, we extend the results to the setting of weak graphs (Theorem 5').

- Section 6 has the proof of the main results, Theorems 1 and 2.
- Appendix A defines a family of energies $E_q^p(f)$ for any $1 \leq p \leq q \leq \infty$ and any PL map f between metric graphs. Theorem 6 gives the most general statement: for any $1 \leq p \leq q \leq \infty$ and homotopy class of maps $[\phi]: G \rightarrow H$ between marked graphs with the appropriate extra structure, we can find $\psi \in [\phi]$ and a tight sequence

$$C \xrightarrow{c} G \xrightarrow{\psi} H \xrightarrow{f} K.$$

(Only a sketch of the proof of Theorem 6 is given.)

Finally, Appendix B relates the elastic networks of this paper and the more well-studied subject of resistor networks and electrical equivalence.

Throughout the paper, we develop the theory behind these concepts, more than is necessary for the proof of the main Theorem 1. For instance, the theory of taut maps in Section 3 is more general than needed for the maps that actually arise in Section 6.

The equality $\text{Emb}[\phi] = \text{SF}_{\text{EL}}[\phi]$ in Theorem 1 is particularly important for applications, in light of the connection between extremal length on surfaces and conformal embeddings [KPT15]. In forthcoming work, we will use this to give criteria for when one degenerating family of surfaces conformally embeds in another. These embeddings of (degenerating) surfaces can in turn be used to give a positive characterization of post-critically finite rational maps among branched self-covers of the sphere [Thu16], giving a converse to an earlier characterization by W. Thurston [DH93].

1.1. Prior and related work. Harmonic maps between manifolds and singular spaces have been studied for a long time, and there is a great deal of work on various cases [GS92, KS93, Wol95, Pic05, inter alia]. In particular, Eells and Fuglede proved that in every homotopy class of maps between suitable (marked) Riemannian polyhedra there is a harmonic representative [EF01, Theorem 11.2], which is a large part of our Theorem 5. Their theory is much more general and more delicate.

The question of when one network is “looser” than another as in Definition 1.9 appears to be new. In a related context, there has been much attention devoted to when two resistor networks are electrically equivalent; see Appendix B for the similarities and differences.

The definition of embedding energy in Definition 1.11 also appears to be new, although it is in a sense dual to Jeremy Kahn’s notion of domination of weighted arc diagrams [Kah06]. Specifically, the criteria mentioned above for embedding degenerating families of surfaces has substantial overlap with Kahn’s work.

Theorem 1 should be thought of as analogous to Teichmüller’s theorem, that in any homotopy class of homeomorphism $[f]: S_1 \rightarrow S_2$ between closed Riemann surfaces there are “best” metrics on S_1 and S_2 that reflect the minimal quasi-conformal stretching. (The length graph K in the statement of Theorem 1 turns out to be G_2 with a different metric.) Indeed, Palmer has an approach to use these techniques to compute Teichmüller maps [Pal15].

Most of the general graph energies in Section A appear to be new, but the energy $E_p^1[c]$ is a power of the p -modulus of the homotopy class, which exists in much greater generality [Fug57].

Many of the results of this paper were announced as part of an earlier research report [Thu16], which also contains many related open problems.

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2. BASIC NOTIONS AND SUB-MULTIPLICATIVITY

As mentioned already, all graphs are assumed to have a linear structure on the edges, metrics are assumed to be linear with respect to this structure, weights are assumed to be piecewise-constant, and maps between graphs are assumed to be PL. All the energies extend naturally to a wider class of maps between graphs; however, the exact wider class of maps depends on the energy, and it is more convenient to stick to PL maps. The downside of working with PL maps is that the existence of minimizers of the energies is not obvious, since the space of PL maps is not locally compact in any reasonable sense. We will prove existence of energy minimizers in each case of interest.

We start by defining our graphs and energies more.

Definition 2.1. If $f: \Gamma_1 \rightarrow \Gamma_2$ is a PL map, the *regular points* of f are the points in Γ_1 that are in the interior of a segment on which f is linear with non-zero derivative, and the *singular points* are all other points, namely vertices of Γ_1 , preimages of vertices of Γ_2 , points where the derivative changes, and segments on which f is constant. Similarly, the *singular values* in Γ_2 are the images of singular points, and the *regular values* are the rest of Γ_2 . There are only finitely many singular values.

Definition 2.2. As in the introduction, a length graph K is a pair (Γ, ℓ) of a graph and an assignment of a length $\ell(e) > 0$ to edge e of Γ ; we call ℓ a *metric* on Γ . In a *weak length*

graph, we weaken the conditions to require $\ell(e) \geq 0$. The space of all weak metrics on Γ is denoted $\mathcal{L}(\Gamma)$, and the subspace of metrics is denoted $\mathcal{L}^+(\Gamma)$. See Definition 4.1 for more on weak metrics. Given a PL map $f: \Gamma_1 \rightarrow (\Gamma_2, \ell)$ to a length graph, we can pull-back the metric on Γ_2 to a weak metric $f^*\ell$ on Γ_1 , assigning to each edge $e \in \text{Edge}(\Gamma_1)$ the total length traced out by $f(e)$. We also write $\ell(f(e))$ for $f^*\ell(e)$.

Definition 2.3. A *weighted graph* is a graph $W = (\Gamma, w)$ with a piecewise-constant weight function $w: \Gamma \rightarrow \mathbb{R}_{\geq 0}$. If w is constant on an edge e , we write $w(e)$ for the weight on e . The space of all weights on Γ that are constant on each edge is denoted $\mathcal{W}(\Gamma)$, and $\mathcal{W}^+(\Gamma)$ is the subspace where the weights are positive.

Definition 2.4. For W a weighted graph, Γ a graph, and $c: W \rightarrow \Gamma$ a PL map, the *multiplicity function* $n_c: \Gamma \rightarrow \mathbb{R}$ is defined at regular values of c by

$$n_c(y) := \sum_{x \in c^{-1}(y)} w(x).$$

This may also be written n_c^w or c_*w if we need to make the dependence on w explicit. We will never care about the value of n_c at non-regular values.

If in addition Γ has more structure we have an energy.

- If Γ is also a weighted graph with its own weights w , the *weight ratio* was defined in Equation (1.25); it is the ℓ^∞ norm of n_c .
- If Γ is an elastic graph with elastic constants α , the *extremal length* of c is

$$(2.5) \quad \text{EL}(c) := \int_{y \in \Gamma} n_c(y)^2 dy,$$

where we view each edge e of G as having a measure of total mass $\alpha(e)$. This generalizes Definition 1.16; $\text{EL}(c)$ is the ℓ^2 norm of n_c , squared.

- If Γ is a length graph with lengths ℓ , the *weighted length* of c was defined in Equation (1.23); by Lemma 2.10 below, it is the ℓ^1 norm of n_c .

In Section 3 we will show that each homotopy class has a *taut map* that simultaneously minimizes $n_c(y)$ for all regular values $y \in \Gamma$. Taut maps are automatically minimizers of WR, EL, and ℓ , independent of which weight, elastic, or length structure we put on Γ . If c is taut, n_c is constant on each edge of Γ and Equation (2.5) reduces to

$$\text{EL}(c) = \sum_{e \in \text{Edge}(G)} n_c(e)^2 \cdot \alpha(e)$$

In addition to these energies, the energies Emb, Dir, and Lip were defined in Section 1. For all of these energies, if we wish to make explicit the dependence of the energy on the geometric structure, we use superscripts for the structure on the domain and subscripts for the structure on the range. Thus we write $\text{WR}_{w_2}^{w_1}$, EL_α^w , ℓ_ℓ^w , $\text{Emb}_{\alpha_2}^{\alpha_1}$, Dir_ℓ^α , or $\text{Lip}_{\ell_2}^{\ell_1}$.

To work with these energies, we give some elementary lemmas, starting with another formula for Dirichlet energy.

Definition 2.6. For G an elastic graph, Γ either a length graph or an elastic graph, and $f: G \rightarrow \Gamma$ a PL map, the *filling function* $\text{Fill}_f: \Gamma \rightarrow \mathbb{R}$ is defined at regular values of f by

$$(2.7) \quad \text{Fill}_f(y) := \sum_{x \in f^{-1}(y)} |f'(x)|.$$

In particular, Equation (1.12) says that $\text{Emb}(f) = \text{ess sup}_y \text{Fill}_f(y)$.

Lemma 2.8. For $f: G \rightarrow K$ a PL map from an elastic graph to a length graph,

$$(2.9) \quad \text{Dir}(f) = \int_{x \in \Gamma} |f'|^2 dx = \int_{y \in K} \text{Fill}_f(y) dy.$$

There is a corresponding formula for ℓ .

Lemma 2.10. For $c: W \rightarrow K$ a PL map from a weighted graph to a length graph,

$$(2.11) \quad \ell(c) = \int_{x \in W} w(x) |c'(x)| dx = \int_{y \in K} n_c(y) dy.$$

Proof of Lemmas 2.8 and 2.10. Change of variables. \square

Lemma 2.12. For $G_1 \xrightarrow{\phi} G_2 \xrightarrow{\psi} \Gamma$ a sequence of PL maps between graphs, where G_1 and G_2 are elastic graphs and Γ is a length graph or an elastic graph, for almost every $z \in \Gamma$,

$$\text{Fill}_{\psi \circ \phi}(z) = \sum_{y \in \psi^{-1}(z)} |\psi'(y)| \text{Fill}_{\phi}(y) \leq \text{Emb}(\phi) \text{Fill}_{\psi}(z).$$

Lemma 2.13. Let $W_1 \xrightarrow{\phi} W_2 \xrightarrow{c} \Gamma$ be a sequence of PL maps, with W_1 and W_2 weighted graphs. Then for almost every $z \in \Gamma$,

$$n_{c \circ \phi}(z) \leq \text{WR}(\phi) n_c(z).$$

Lemma 2.14. Let $\Gamma \xrightarrow{f} K_1 \xrightarrow{\phi} K_2$ be a sequence of PL maps, with K_1 and K_2 length graphs. Then for almost every $x \in \Gamma$,

$$|(\phi \circ f)'(x)| \leq |f'(x)| \text{Lip}(\phi).$$

Proof of Lemmas 2.12, 2.13, and 2.14. Immediate from the definitions. \square

We now turn to sub-multiplicativity, Equations (1.27) and (1.28). For concreteness (and because the square roots get a little confusing), we list all 10 cases individually.

Proposition 2.15. The energies from (1.22) are sub-multiplicative, in the following sense. For $i \in \{1, 2, 3\}$, let W_i be marked weighted graphs, G_i be marked elastic graphs, and K_i be marked length graphs. Then, if we are given marked PL maps between these spaces as specified on each line, we have the given inequality.

$$(2.16) \quad W_1 \xrightarrow{\phi} W_2 \xrightarrow{\psi} W_3: \quad \text{WR}(\psi \circ \phi) \leq \text{WR}(\phi) \text{WR}(\psi)$$

$$(2.17) \quad W_1 \xrightarrow{\phi} W_2 \xrightarrow{c} G: \quad \text{EL}(c \circ \phi) \leq \text{WR}(\phi)^2 \text{EL}(c)$$

$$(2.18) \quad W_1 \xrightarrow{\phi} W_2 \xrightarrow{f} K: \quad \ell(f \circ \phi) \leq \text{WR}(\phi) \ell(f)$$

$$(2.19) \quad W \xrightarrow{c} G_1 \xrightarrow{\phi} G_2: \quad \text{EL}(\phi \circ c) \leq \text{EL}(c) \text{Emb}(\phi)$$

$$(2.20) \quad W \xrightarrow{c} G \xrightarrow{f} K: \quad \ell(f \circ c)^2 \leq \text{EL}(c) \text{Dir}(f)$$

$$(2.21) \quad W \xrightarrow{c} K_1 \xrightarrow{\phi} K_2: \quad \ell(\phi \circ c) \leq \ell(c) \text{Lip}(\phi)$$

$$(2.22) \quad G_1 \xrightarrow{\phi} G_2 \xrightarrow{\psi} G_3: \quad \text{Emb}(\psi \circ \phi) \leq \text{Emb}(\phi) \text{Emb}(\psi)$$

$$(2.23) \quad G_1 \xrightarrow{\phi} G_2 \xrightarrow{f} K: \quad \text{Dir}(f \circ \phi) \leq \text{Emb}(\phi) \text{Dir}(f)$$

$$(2.24) \quad G \xrightarrow{f} K_1 \xrightarrow{\phi} K_2: \quad \text{Dir}(\phi \circ f) \leq \text{Dir}(f) \text{Lip}(\phi)^2$$

$$(2.25) \quad K_1 \xrightarrow{\phi} K_2 \xrightarrow{\psi} K_3: \quad \text{Lip}(\psi \circ \phi) \leq \text{Lip}(\phi) \text{Lip}(\psi).$$

(When there is only one graph of a particular type, we omit the subscript.) Each inequality still holds if we take homotopy classes on both sides.

Proof. Equations (2.16), (2.17), and (2.18) follow from Lemma 2.13 and Equations (1.25), (2.5), and (2.11), respectively. Equations (2.21), (2.24), and (2.25) follow from Lemma 2.14 and Equations (2.11), (1.5), and (1.24), respectively.

For Equation (2.19), we have

$$\begin{aligned} \text{EL}(\phi \circ c) &= \int_{z \in G_2} n_{\phi \circ c}(y)^2 dz \\ &= \int_{z \in G_2} \left(\sum_{\phi(y)=z} n_c(y) \right)^2 dz \\ &\leq \int_{z \in G_2} \left(\sum_{\phi(y)=z} n_c(y)^2 / |\phi'(y)| \right) \left(\sum_{\phi(y)=z} |\phi'(y)| \right) dz \\ &\leq \text{Emb}(\phi) \int_{y \in G_1} n_c(y)^2 dy \\ &= \text{Emb}(\phi) \text{EL}(c), \end{aligned}$$

using the definition of EL; the equality $n_{\phi \circ c}(y) = \sum_{\phi(x)=y} n_c(x)$ at regular values; the Cauchy-Schwarz inequality; the definition of $\text{Emb}(\phi)$ and a change of variables between G_2 and G_1 ; and the definition of EL again.

For Equation (2.20), we have

$$\begin{aligned} \ell(f \circ c)^2 &= \left(\int_{z \in K} n_{f \circ c}(z) dz \right)^2 \\ &= \left(\int_{y \in G} n_c(y) |f'(y)| dy \right)^2 \\ &\leq \left(\int_{y \in G} n_c(y)^2 dy \right) \left(\int_{y \in G} |f'(y)|^2 dy \right) \\ &= \text{EL}(c) \text{Dir}(f), \end{aligned}$$

using Lemma (2.10), a change of variables from K to G , the Cauchy-Schwarz inequality, and the definitions of EL and Dir.

Equation (2.22) follows from Lemma 2.12.

For Equation (2.23), we have

$$\begin{aligned}
\text{Dir}(f \circ \phi) &= \int_{z \in K} \text{Fill}_{f \circ \phi}(z) dz \\
&\leq \int_{z \in K} \text{Fill}_f(z) \left(\max_{f(y)=z} \text{Fill}_\phi(y) \right) dz \\
&\leq \int_{z \in K} \text{Fill}_f(z) \text{Emb}(\phi) dz \\
&= \text{Emb}(\phi) \text{Dir}(f),
\end{aligned}$$

using Lemma 2.8, Lemma 2.12, the definition of $\text{Emb}(\phi)$, and Lemma 2.8 again.

For each of these equations, to replace concrete functions by homotopy classes, take representatives f, g of the two homotopy classes whose energy is within a factor of ε of the optimal value, in the sense that $E_q^p[f] \leq E_q^p(f) \leq E_q^p[f](1 + \varepsilon)$, and similarly for g . Then

$$E_r^p[g \circ f] \leq E_r^p(g \circ f) \leq E_q^p(f) E_r^q(g) \leq E_q^p[f] E_r^q[g] (1 + \varepsilon)^2.$$

Since ε can be chosen as small as desired, we are done. \square

We now have one direction of Theorem 1.

Corollary 2.26. *For any homotopy class $[\phi]: G_1 \rightarrow G_2$ of maps between marked elastic graphs, $\text{SF}_{\text{Dir}}[\phi] \leq \text{Emb}[\phi]$ and $\text{SF}_{\text{EL}}[\phi] \leq \text{Emb}[\phi]$.*

Proof. For any marked length graph K and homotopy class $[f]: G_2 \rightarrow K$, by the homotopy version of Equation (2.23) we have

$$\frac{\text{Dir}[f \circ \phi]}{\text{Dir}[f]} \leq \frac{\text{Dir}[f] \text{Emb}[\phi]}{\text{Dir}[f]} = \text{Emb}[\phi].$$

Since K and $[f]$ were arbitrary, it follows that $\text{SF}_{\text{Dir}}[\phi] \leq \text{Emb}[\phi]$. The argument that $\text{SF}_{\text{EL}}[\phi] \leq \text{Emb}[\phi]$ is exactly parallel. \square

3. REDUCED AND TAUT MAPS

We next turn to notions of efficiency of maps between graphs. We first have a weak notion depending on no extra structure (“reduced”), and then develop more a notion that depends on a weight structure (“taut”).

3.1. Reduced maps. We work by analogy with the standard notion of a reduced (cyclic) word in a group.

Definition 3.1. A map $f: \Gamma_1 \rightarrow \Gamma_2$ between marked graphs is *edge-reduced* if, on each edge e of Γ_1 , f is either constant or has a perturbation that is locally injective.

Definition 3.2. For a graph Γ and point $x \in \Gamma$, a *direction* d at x is a germ of a PL map from $\mathbb{R}_{\geq 0}$ to Γ starting at x , considered up to PL reparametrization. There is always a *zero* direction, the germ of a constant map. Points on edges of Γ have two non-zero directions, and vertices have as many non-zero directions as their valence. If $f: \Gamma_1 \rightarrow \Gamma_2$ is a PL map and d is a direction at $x \in \Gamma_1$, then $f(d)$ is a direction at $f(x)$.

Definition 3.3. A PL map $f: \Gamma_1 \rightarrow \Gamma_2$ between marked graphs is *strictly reduced* if it is locally injective on the edges of Γ_1 and, at each unmarked vertex v of Γ_1 , there are directions d_1 and d_2 at v so that $f(d_1)$ and $f(d_2)$ are distinct and non-zero.

More generally, pick $y \in \Gamma_2$ and let $Z \subset \Gamma_1$ be a component of $f^{-1}(y)$. A *direction* from Z is a point $x \in Z$ and a direction d from x that points out of Z (so that $f(d)$ is non-zero). Then Z is a *dead end* for f if Z has no marked point and there is exactly one direction in

$$D(Z) := \{ f(d) \mid d \text{ a direction from } Z \}.$$

(If $D(Z) = \emptyset$, then Z is necessarily an entire connected component of Γ_1 . We do not count this as a dead end.) We say that f is *reduced* if it has no dead ends. If f is not reduced, there is a natural *reduction* operation at a dead end Z , where we modify f by pulling the image of Z in the direction $D(Z)$ until it hits a vertex of Γ_2 or the boundary of a domain of linearity.

Proposition 3.4. *If $f: \Gamma_1 \rightarrow \Gamma_2$ is any map between marked graphs, then repeated reduction makes f into a reduced map. In particular, there is a reduced map in every homotopy class.*

Proof. Proceed by induction on the number of linear segments of f with non-zero derivative. This is reduced by each reduction operation. \square

Proposition 3.5. *For $p, q \in \{1, 2, \infty\}$ with $1 \leq p \leq q \leq \infty$, reduction does not increase E_q^p , and strictly decreases E_q^p if $p < q$.*

Proof. Clear from the definitions. \square

As a result of Propositions 3.4 and 3.5, when looking for energy minimizers we can restrict our attention to reduced maps.

3.2. Taut maps and flows. We now add some more structure, and correspondingly get more restrictive conditions on energy-minimizing maps. We will consider maps from weighted graphs, and in particular how to minimize E_q^1 for any q .

Definition 3.6. Let $c: W \rightarrow \Gamma$ be a PL map from a marked weighted graph W to a marked graph Γ . We defined the multiplicity $n_c: \Gamma \rightarrow \mathbb{R}_{\geq 0}$ in Definition 2.4. To define a multiplicity for homotopy classes, for y in the interior of an edge of Γ set

$$n_{[c]}(y) := \inf_{d \in [c]} n_d(y).$$

The infimum is taken over maps d for which $n_d(y)$ is defined. Then $n_{[c]}(y)$ depends only on the edge containing y . We say that c is *taut* if $n_c = n_{[c]}$ almost everywhere. We say that c is *locally taut* if every regular value $y \in \Gamma$ has a regular neighborhood N so that n_c cannot be reduced by homotopy of c on $c^{-1}(N)$. (A *regular neighborhood* of y is a tubular neighborhood of y that has no singular points except possibly at y . In particular, it has no other vertices of Γ .)

It will require work to show that taut maps exist, but if they do exist they have good properties.

Lemma 3.7. *A taut map from a positive weighted graph is reduced.*

Proof. Reduction reduces n_c . \square

Proposition 3.8. *Let $c: W \rightarrow \Gamma$ be a taut map from a marked weighted graph W to a marked graph Γ . If Γ is weighted, then $\text{WR}(c)$ is minimal in $[c]$. If Γ has an elastic structure, then $\text{EL}(c)$ is minimal in $[c]$. If Γ has a length structure, then $\ell(c)$ is minimal in $[c]$.*

Proof. Clear from the definitions, since the energies are monotonic in n_c . When Γ is a length graph, we use Lemma 2.10. \square

Example 3.9. Let W be a marked weighted graph with two marked vertices s and t . Let I be the interval $[-1, 1]$ with the endpoints marked, and let $f: W \rightarrow I$ be a map with $f(s) = -1$ and $f(t) = 1$. Then f is taut iff each edge is mapped monotonically and, for each regular value $y \in I$, the set of edges containing $f^{-1}(y)$ is a *minimal cut-set* for W , a set of edges of W that separates s from t and has minimal weight among all such sets.

The max-flow/min-cut theorem says that in the context of Example 3.9 the total weight of a minimal cut-set for W is equal to the maximum flow from s to t through the edges of t . We will show that taut maps exist, and that they satisfy a generalization of the max-flow/min-cut theorem; see Theorem 3 and Corollary 3.29 below. To give the strongest statement, we make some definitions and preliminary lemmas first.

Lemma 3.10. *A marked weighted multi-curve $c: C \rightarrow \Gamma$ on a marked graph Γ , with weights that are positive and constant on the components of C , is taut iff it is reduced.*

Proof. One direction is Lemma 3.7. For the other direction, use the standard fact that, given a non-trivial free homotopy class of maps from S^1 to Γ , a strictly reduced representative is unique up to reparametrization of the domain (which does not change n_c). The same is true for homotopy classes of maps from I to Γ relative to the endpoints. \square

Definition 3.11. If $c: W_1 \rightarrow W_2$ is a PL map between weight graphs, we say that W_2 *carries* (W_1, c) if $\text{WR}(c) \leq 1$. If W_2 carries (W_1, c) , then we say that a point y of W_2 is *saturated* if $n_c(y) = w(y)$. Similarly, a subset of a weighted graph is saturated if almost every point in it is saturated.

One notion of a “flow” on a weighted graph W is a taut weighted multi-curve carried by W . These curves are a little awkward to work with directly. As such, we will also define and work with a more general type of flow from *train tracks*.

Definition 3.12. A sequence of non-negative numbers $(x_i)_{i=1}^k$ is said to satisfy the *triangle inequalities* if, for each i , x_i is no larger than the sum of the remaining numbers:

$$(3.13) \quad x_i \leq \sum_{\substack{1 \leq j \leq k \\ i \neq j}} x_j.$$

This implies that $k \neq 1$, and if $k = 2$ then $x_1 = x_2$. If $k \geq 3$ and one of these inequalities is an equality, then there is exactly one i so that x_i is equal to the sum of the remainder. That x_i is said to *dominate* the rest.

Definition 3.14. A *train-track structure* τ on a graph Γ is, for each point x of Γ , a partition of the non-zero directions from x into equivalence classes, called the *gates* at x , with at least two gates at each unmarked point.

If Γ is weighted, the *weight* of a gate is the sum of the weights of the edges corresponding to the directions that make it up. A *weighted train track* T is a tuple (Γ, w, τ) of a marked weighted graph (Γ, w) with a train-track structure τ so that, at each unmarked vertex, the

weights of the gates satisfy the triangle inequalities. (If we don't want to impose the triangle inequalities, we may refer to a weighted graph with a train-track structure.) If the weight of one gate g at a vertex of a weighted train track dominates the others, we can *smooth* the vertex, changing the train-track structure so that there are only two gates, one with the directions from g and one with all the other directions.

A graph Γ with no unmarked univalent ends has a *discrete* train-track structure δ_Γ , in which two different directions are never equivalent. By default we use the discrete train-track structure on a graph. A weighted graph for which the discrete train-track structure satisfies the triangle inequalities is said to be *balanced*.

If $f: (\Gamma_1, \tau_1) \rightarrow (\Gamma_2, \tau_2)$ is a map between train tracks that is locally injective on the edges of Γ_1 , we say that f is a *train-track map* if, for each vertex v of Γ_1 and directions d_1 and d_2 at v ,

$$(3.15) \quad (d_1 \sim_{\tau_1} d_2) \iff (f(d_1) \sim_{\tau_2} f(d_2)).$$

More generally, we say that f is a train-track map if it has arbitrarily small perturbations f_ε so that f_ε is locally injective on the edges of Γ_1 and satisfies Equation (3.15). In particular, a curve $c: C \rightarrow \Gamma$ is a train-track map iff at each point it passes through the incoming and outgoing directions are in different gates.

If $f: \Gamma_1 \rightarrow \Gamma_2$ is a strictly reduced map, the *train track* of f is the unique train-track structure $\tau(f)$ on Γ_1 so that f is a train-track map with respect to the train track structures $\tau(f)$ on Γ_1 and δ_{Γ_2} on Γ_2 . Concretely,

$$(d_1 \sim_{\tau(f)} d_2) \iff (f(d_1) = f(d_2)).$$

A composition of train-track maps is a train-track map.

We can now state the main goal of this section.

Theorem 3. *Let $f: W \rightarrow \Gamma$ be a PL map from a marked weighted graph to a marked graph. Then there is a taut map in $[f]$. Furthermore, the following conditions are equivalent.*

- (1) *The map f is taut.*
- (2) *The map f is locally taut.*
- (3) *The graph W carries a marked weighted train track $t: T \rightarrow W$, so that $f \circ t$ is a train-track map (with respect to the discrete train track structure on Γ) and $n_{f \circ t} = n_t$. We may choose (T, t) so that T is a subgraph of W .*
- (4) *The graph W carries a marked weighted multi-curve $c: C \rightarrow W$ so that $f \circ c$ is taut and $n_{f \circ c} = n_f$.*

Furthermore, in conditions (3) and (4), c and t , respectively, saturate every edge of W on which f is not constant.

As a preliminary step towards Theorem 3, we relate train tracks and multi-curves.

Proposition 3.16. *Any weighted train track $T = (\Gamma, w, \tau)$ carries a marked weighted multi-curve (C, c) that saturates T and so that $c: C \rightarrow T$ is a train-track map. We can choose (C, c) so that each component of C runs over each edge of Γ at most twice.*

Proof. Let $T' \subset T$ be the sub-train-track of edges of non-zero weight. Let T'' be T' modified by smoothing all vertices in which one gate dominates the others. Let $\text{Yard}(T'') \subset \text{Vert}(T'')$ be the set of marked vertices of T'' with at least three gates. We will proceed by induction on $|\text{Edge}(T'')| + |\text{Yard}(T'')|$.

In the base case, T' and T'' are empty.

Otherwise, choose any oriented edge \vec{e}_0 of T'' , and find a path (forward and backwards) from \vec{e}_0 within T'' , always making turns between different gates. Since there are at least two gates at each marked vertex, we can always find a successor edge for \vec{e}_i , unless we hit a marked vertex. Since there are only finitely many oriented edges in T , we will eventually either find a path of edges between marked points or see a repeat and find a cyclic loop of edges.

Consider the marked curve (C_1, c_1) that runs over the cycle or path. For $\varepsilon > 0$, let $w_\varepsilon(e) := w(e) - \varepsilon n_{c_1}(e)$. Then for ε sufficiently small, w_ε gives a weighted train-track structure on T'' :

- On an edge e , $w(e) > 0$ by construction of T' so $w_\varepsilon(e) \geq 0$; and
- At a vertex v , the construction of T'' ensures that the triangle inequalities at v continue to hold in w_ε .

Let ε_1 be the maximum value of ε so that $(\Gamma, w_{\varepsilon_1}, \tau)$ is a weighted train track. Let $w_1 = w_{\varepsilon_1}$ and $T_1 = (\Gamma, w_1, \tau)$. Consider the derived train tracks T'_1 (deleting edges of weight 0) and T''_1 (smoothing dominating vertices). By the choice of ε_1 , there is either

- an edge e of T'' with $w_1(e) = 0$ but $w(e) \neq 0$, or
- a vertex v with a gate g so that $w_1(g)$ dominates the other gates but $w(g)$ does not.

In the first case, $|\text{Edge}(T'_1)| < |\text{Edge}(T'')|$. In the second case, $|\text{Yard}(T'_1)| < |\text{Yard}(T'')|$. In either case, by induction T_1 carries a marked weighted multi-curve (C_2, c_2) that saturates T_1 . Then $c_2 \sqcup c_1: C_2 \sqcup C_1 \rightarrow T$ is the desired multi-curve from the statement, where C_1 has weight ε_1 .

If we make a cyclic loop as soon as we see a repeated oriented edge, the components of C run over each (unoriented) edge at most twice. \square

Lemma 3.17. *If $f: W \rightarrow \Gamma$ is a map from a marked weighted graph to a marked graph and (C, c) is a marked weighted multi-curve carried by W so that $f \circ c$ is taut and $n_{f \circ c} = n_f$, then f is taut.*

Proof. If $g: W \rightarrow \Gamma$ is any other map in $[f]$, then

$$n_g \geq n_{g \circ c} \geq n_{[f \circ c]} = n_{f \circ c} = n_f,$$

using the fact that W carries c (and Lemma 2.13), the definition of $n_{[f \circ c]}$ as an infimum over the homotopy class, and the hypotheses. \square

Lemma 3.18. *A train-track map from a marked weighted train track is taut.*

Proof. Immediate from Proposition 3.16 and Lemma 3.17. \square

3.3. Local models. To prove Theorem 3, we first analyze the situation locally in a regular neighborhood of a singular value. This reduces to studying maps from a graph with k marked points to a k -leg star graph.

Let Star_k be the star graph with k legs, with a central vertex s_* , marked endpoints s_1, \dots, s_k , and edges $[s_i, s_*]$. A k -marked graph is a graph with k marked vertices $\{v_i\}_{i=1}^k$ (in order). There is a canonical homotopy class of marked maps from a k -marked graph to Star_k , taking v_i to s_i . There is an analogue of Theorem 3 in this context.

Proposition 3.19. *Let W be a k -marked weighted graph. Then there is a taut map in the canonical homotopy class of maps to Star_k . Furthermore, the following conditions are equivalent.*

- (1) The map f is taut.
- (2) The graph W carries a marked weighted train track (T, t) so that $f \circ t$ is a train-track map and $n_{f \circ t} = n_f$.
- (3) The graph W carries a marked weighted curve (C, c) so that $f \circ c$ is taut and $n_{f \circ c} = n_f$.

Definition 3.20. A *cut* S of a graph Γ is a partition of the vertices of the graph into two disjoint subsets: $\text{Vert}(\Gamma) = S \sqcup \bar{S}$. The corresponding *cut-set* $c(S) = c(\bar{S})$ is the set of edges that connect S to \bar{S} . If Γ has weights w , the *weight* of the cut is $w(S) := \sum_{e \in c(S)} w(e)$.

Two cuts S_1 and S_2 are *nested* if they are disjoint or one is contained in the other : $S_1 \cap S_2 = \emptyset$, $S_1 \subset S_2$, or $S_2 \subset S_1$. (It doesn't matter if we replace S_1 with \bar{S}_1 and/or replace S_2 with \bar{S}_2 .)

Let W be a k -marked weighted graph. A v_i -cut is a subset $S_i \subset \text{Vert}(G)$ so that $S_i \cap \{v_1, \dots, v_k\} = \{v_i\}$. A *vertex cut* is a v_i -cut for some i . A *minimal* v_i -cut is one with minimal weight. Let $\text{mincut}_i(w)$ be the weight of a minimal v_i -cut.

Lemma 3.21. If S_1 and S_2 are two cuts on the same weighted graph, then $w(S_1) + w(S_2) \geq w(S_1 \cap S_2) + w(S_1 \cup S_2)$.

Proof. This is the standard sub-modular property of cuts, and is easy to prove. \square

Lemma 3.22. If W is a k -marked weighted graph and S_i and S'_i are two minimal v_i -cuts, then $S_i \cap S'_i$ and $S_i \cup S'_i$ are also minimal v_i -cuts.

Proof. The sets $S_i \cap S'_i$ and $S_i \cup S'_i$ are v_i -cuts, so by minimality they all have weight at least as large as $\text{mincut}_i(w) = w(S_i) = w(S'_i)$. Lemma 3.21 gives the other inequality. \square

Lemma 3.23. If W is a k -marked weighted graph, S_i is a minimal v_i -cut, and S_j is a minimal v_j cut for $j \neq i$, then $S_i \setminus S_j$ is also a minimal v_i -cut and $S_j \setminus S_i$ is also a minimal v_j -cut.

Proof. $S_i \setminus S_j$ is a v_i -cut and $S_j \setminus S_i$ is a v_j -cut. By minimality, their weights are at least as large as $\text{mincut}_i(w)$ and $\text{mincut}_j(w)$, respectively. Applying Lemma 3.21 to S_i and \bar{S}_j gives the other inequalities. \square

Definition 3.24. If W is a k -marked weighted graph, we say that an edge of W is *slack* if it has non-zero weight and is not contained in any minimal vertex cut. W is *minimal* if it has no slack edges.

See Figure 2 for the next two lemmas.

Lemma 3.25. If $W = (\Gamma, w)$ is a weighted graph with k marked vertices, then there is a set of weights $w_0 \leq w$ on Γ so that $W_0 = (\Gamma, w_0)$ is minimal and so that for all i , $\text{mincut}_i(w) = \text{mincut}_i(w_0)$.

Proof. We proceed by induction on the number of slack edges. If W is not minimal, pick any slack edge e_0 of W . For $0 \leq k \leq w(e_0)$, define a modified set of weights by

$$w\{e_0 \mapsto k\}(e) := \begin{cases} k & e = e_0 \\ w(e) & \text{otherwise.} \end{cases}$$

Let k_0 be minimal value so that, for all i , $\text{mincut}_i(w\{e_0 \mapsto k_0\}) = \text{mincut}_i(w)$. Then $k < w(e_0)$ and e is not slack with respect to $w\{e_0 \mapsto k_0\}$. By induction we can find weights $w_0 \leq w\{e_0 \mapsto k_0\} \leq w$ on Γ so that (Γ, w_0) is minimal. \square

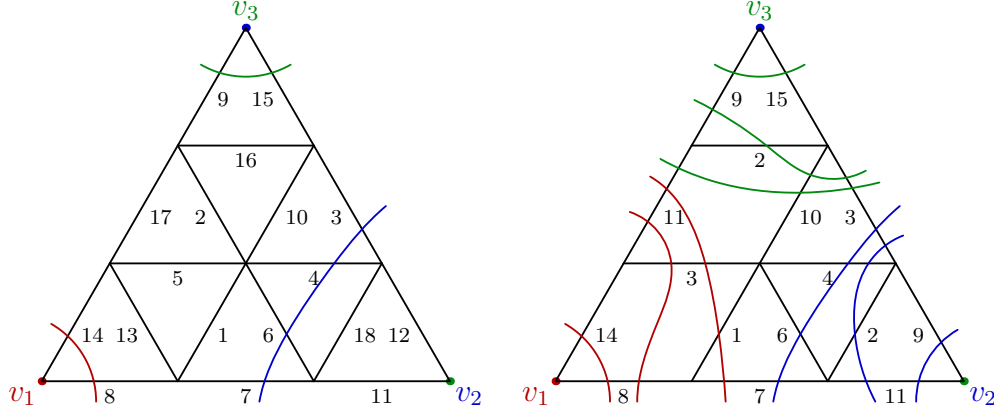


FIGURE 2. Finding minimum cuts and maximum flows near a vertex. Left: a weighted graph W with three marked vertices, with the weights indicated. Minimal cuts that separate the vertices are marked; these are used to construct a taut map from W to Star_3 . Right: a weighted graph W_0 carried by W , with minimal weights that have the same mincut_i for each i . The minimal cuts on W have been extended to a complete nested family of minimal cuts. These are used to construct a train track carried by W .

Lemma 3.26. *Let $W = (\Gamma, w)$ be a k -marked weighted graph that is minimal and let $\mathcal{S} \subset \mathcal{P}(\mathcal{P}(\text{Vert}(W)))$ be a nested set of minimal vertex cuts. Then there is a nested set of minimal vertex cuts $\mathcal{T} \supset \mathcal{S}$ so that every edge of W with non-zero weight is contained in $c(S)$ for some $S \in \mathcal{T}$.*

(Here $\mathcal{P}(X)$ is the power set of X .)

Proof. We proceed by induction on the size of $\text{Edge}(W) \setminus \bigcup_{S \in \mathcal{S}} c(S)$. If there is at least one edge e_1 in this set with non-zero weight, it is contained in some minimal vertex cut-set $c(S_1)$ by minimality of W . Then, by repeatedly applying Lemmas 3.22 and 3.23, we can find another minimal vertex cut S'_1 so that $e \in c(S'_1)$ and S'_1 is nested with respect to all $S \in \mathcal{S}$. By induction, $\mathcal{S} \cup \{S'_1\}$ can be completed to the desired set \mathcal{T} . \square

Proof of Proposition 3.19. We first prove the existence of a taut map. For each i , pick a minimal v_i -cut S_i , and let $S'_i = S_i \setminus \bigcup_{j \neq i} S_j$. By Lemma 3.23, $\{S'_i\}$ is a nested collection of minimal v_i -cuts. Define $f: W \rightarrow \text{Star}_k$ by mapping all vertices in S'_i to s_i , all vertices in $\text{Vert}(G) \setminus \bigcup_i S'_i$ to s_* , and all edges to reduced paths connecting their endpoints. (Thus f is constant on any edge not in any $c(S'_i)$.) Then f is taut since the S'_i are minimal.

Proposition 3.16 and Lemma 3.10 tell us that (2) implies (3), and Lemma 3.17 tells us that (3) implies (1), so it remains to prove that (1) implies (2).

Now suppose that f is taut. For each regular value $y \in [s_*, s_i] \subset \text{Star}_k$, consider the v_i -cut $S_y = \{v \in \text{Vert}(\Gamma) \mid f(v) \in [s_i, y]\}$. Then S_y is a minimal vertex cut, since f is taut. Furthermore, if z is another regular value of f on any edge of Star_k , then S_y and S_z are nested. Let $\mathcal{S} = \{S_y \mid y \in \text{Star}_k, y \text{ regular}\}$, considered as a nested system of minimal cuts on W .

Use Lemma 3.25 to find weights $w_0 \leq w$ so that (Γ, w_0) is minimal. Let $\Gamma_0 \subset \Gamma$ consist of the edges e with $w_0(e) \neq 0$, let $W_0 = (\Gamma_0, w_0)$, and let $t: W_0 \hookrightarrow W$ be the inclusion. By

Lemma 3.26, \mathcal{S} can be completed to a nested system of cuts \mathcal{T} on W_0 so that every edge is in at least one cut-set.

For each i , let the distinct v_i -cuts in \mathcal{T} be

$$\{v_i\} = S_{i,0} \subsetneq S_{i,1} \subsetneq \cdots \subsetneq S_{i,n_i}.$$

For each i and for $0 \leq j \leq n_i$, pick distinct points $x_{i,j} \in [s_i, s_*) \subset \text{Star}_k$ with $x_{i,0} = s_i$ and with the $x_{i,j}$ in order. Then define a map $g: W_0 \rightarrow \text{Star}_k$ on vertices by

$$g(v) = \begin{cases} s_i & v = v_i \\ x_{i,j} & v \in S_{i,j} \setminus S_{i,j-1} \\ s_* & v \in \text{Vert}(W_0) \setminus \bigcup_{S \in \mathcal{T}} S. \end{cases}$$

(These cases are exclusive since \mathcal{T} is nested.) On an edge e of W_0 , define $g(e)$ to be a reduced path connecting its endpoints. Since \mathcal{T} is a complete set of cuts, no edge maps to a single point. Let $\tau_0 = \tau(g)$. Then (W_0, τ_0) is a weighted train track: the relevant triangle (in)equalities are implied by the fact that the $S_{i,j}$ are all minimal cuts.

By an appropriate choice of the $x_{i,j}$, the train-track map g can be made to be an arbitrarily small perturbation of $f \circ t$, so $f \circ t$ is also a train-track map. If e is an edge on which the original map f is not constant, then e is not slack and $w(e) = w_0(e)$, so $n_{f \circ t} = n_f$ as desired. \square

3.4. General flows. We now use these local models to prove Theorem 3.

Definition 3.27. Let $f: W \rightarrow \Gamma$ be a PL map from a marked weighted graph W to a marked graph Γ . Pick $y \in \Gamma$ and let $N \subset \Gamma$ be a closed regular neighborhood of y . Then the *local model for f at y* is the map

$$f^{-1}(N)/\sim \longrightarrow N,$$

where

- N is considered as a marked graph (equivalent to Star_k) with marked points at ∂N ;
- \sim is the equivalence relation that identifies two points in $f^{-1}(\partial N)$ if they map to the same point in ∂N ; and
- $f^{-1}(N)/\sim$ is considered as a k -marked weighted graph, with weights inherited from W .

Lemma 3.28. Let $[f]: W \rightarrow \Gamma$ be a homotopy class of maps from a marked weighted graph W to a marked graph Γ . Then there is a locally taut map in $[f]$.

Proof. Suppose f is any PL representative for its homotopy class. We can modify f to send vertices to vertices without increasing n_f , as follows. For each edge e of Γ , look at the division of e into intervals according to the value of n_f . Pick one of these intervals on which n_f is minimal (among the values of n_f that appear), and spread out this interval by a homotopy until it covers e . This gives us an initial map f_0 .

If f_0 is not locally taut, then there is some vertex v of Γ so that the local model of f_0 at v is not taut. By Proposition 3.19, there is a taut map in the homotopy class of the local model. Let f'_0 be f_0 with the map replaced by its taut model near v . There is at least one edge e_0 of Γ with a segment near v on which $n_{f'_0} < n_{f_0}$. Homotop f'_0 as above, spreading out segments of minimal multiplicity, to construct a map f_1 that sends vertices to vertices with $n_{f_1} \leq n_{f_0}$ everywhere and $n_{f_1}(e_0) < n_{f_0}(e_0)$.

If f_1 is not locally taut, repeat the process, with f_1 in place of f_0 . Our initial representative f_0 gives an upper bound on $n_{f_i}(e)$ for all e . There are only finitely many linear combinations

of the non-zero weights of edges of W to get a value less than this bound. At each step n_f strictly decreases on at least one edge, so the process terminates in finite time. \square

Proof of Theorem 3. We first prove the equivalence of the four conditions on taut maps. (1) implies (2) is obvious.

To see that (2) implies (3), suppose f is locally taut. Then for each singular value $y \in \Gamma$, the local model for f at y carries a weighted train track compatible with f by Proposition 3.19. Define a weighted train track T and map $t: T \rightarrow W$ by assembling these local models following the pattern of W , leaving W unchanged outside of the local models. Then (T, t) is the desired train track carried by W , with T a subgraph of W .

Proposition 3.16 shows that (3) implies (4).

Lemma 3.17 shows that (4) implies (1).

Now if $[f]$ is any homotopy class, by Lemma 3.28 there is a locally taut element of $[f]$, which by the above equivalences is also globally taut.

The statements about saturation follow immediately. \square

3.5. Connection to max-flow/min-cut. We can connect Theorem 3 and Proposition 3.19 to a statement that looks more like the classical max-flow/min-cut problem (Example 3.9). For simplicity, we restrict to the local setting of Proposition 3.19. Given a graph Γ with k marked points $\{v_i\}_{i=1}^k$, define a *flow* from v_i to v_j to be a marked weighted multi-curve (C, c) on Γ , with each component a marked interval with one endpoint mapping to v_i and the other mapping to v_j . Such a flow has multiplicities n_c on each edge as usual, as well as a total weight $w(c)$, the sum of the weights of all components of C .

Corollary 3.29. *Let W be a weighted graph with k marked points $\{v_i\}_{i=1}^k$. Then we can find*

- *for $1 \leq i \leq k$, a v_i -cut S_i , and*
- *for $1 \leq i < j \leq k$, a flow $c_{ij} = c_{ji}$ from v_i to v_j*

so that

- *the collection of all flows c_{ij} is carried by W : for each edge e of Γ ,*

$$(3.30) \quad \sum_{i < j} n_{c_{ij}}(e) \leq w(e),$$

- *the total flow into a vertex equals the weight of the cut: for each i ,*

$$(3.31) \quad \sum_{j \neq i} w(c_{ij}) = w(S_i).$$

In this situation, for each i , $w(S_i)$ is minimal and $\sum_{j \neq i} w(c_{ij})$ is maximal.

Proof. Let f be a taut map in the canonical homotopy class of maps from W to Star_k , as given by Proposition 3.19. By that proposition, W carries a marked weighted multi-curve (C, c) so that $f \circ c$ is taut and $n_{f \circ c} = n_f$. The only non-trivial homotopy classes of marked multi-curves on Star_k are given by paths connecting s_i to s_j for some $i \neq j$. Let the flow c_{ij} be the flow given by those components of (C, c) that connect v_i to v_j . Then Equation (3.30) says that W carries (C, c) .

On the other hand, for each i , let y_i be a regular value on $[s_*, s_i]$, and let S_i be S_{y_i} as defined in the proof of Proposition 3.19. Then the equality $n_{f \circ c}(y) = n_f(y)$ is equivalent to Equation (3.31). Minimality of $w(S_i)$ and maximality of $\sum_{j \neq i} w(c_{ij})$ both follow from tautness, and indeed are easy to deduce from Equation (3.31). \square

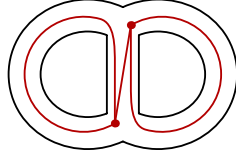


FIGURE 3. A map from an eyeglass graph to a theta graph that is reduced but not strongly reduced.

3.6. Strongly reduced maps. We introduce a stronger notion of “reduced” for maps between general marked graphs without a weight structure.

Definition 3.32. A map $f: \Gamma_1 \rightarrow \Gamma_2$ between marked graphs is *strongly reduced* if it is taut for some choice of positive weights on Γ_1 . See Figure 3 for a non-example.

Lemma 3.33. *Let $f: \Gamma_1 \rightarrow \Gamma_2$ be an edge-reduced map. Then f is strongly reduced iff for each edge e of Γ_1 , either f is constant on e or there is a reduced curve (C, c) on Γ_1 that runs over e and so that $f \circ c$ is also reduced. Furthermore, the curves can be chosen to run over each edge of Γ_1 at most twice.*

Proof. The first part is a consequence of the equivalence of conditions (1) and (4) in Theorem 3. For the second part, use Proposition 3.16. \square

Definition 3.34. Given an edge-reduced PL map $f: \Gamma_1 \rightarrow \Gamma_2$ between marked graphs, the *combinatorial type* of f consists of the following discrete data.

- (1) For each vertex v of Γ_1 , record whether $f(v)$ is on an vertex or in the interior of an edge of Γ_2 , and which vertex or edge it lies on.
- (2) For each oriented edge \vec{e} of Γ_1 , record the *edge-path* of $f(\vec{e})$, the reduced sequence of oriented edges of Γ_2 that $f(\vec{e})$ travels over. This edge-path may start and/or end with a partial edge, depending on whether the endpoints of \vec{e} map to vertices or edges. There are degenerate cases when \vec{e} maps to a single vertex or stays within the interior of a single edge.

The combinatorial type of f determines the homotopy class of $[f]$.

Proposition 3.35. *Let $[f]: \Gamma_1 \rightarrow \Gamma_2$ be a homotopy class of maps between marked graphs. Then there are only finitely many combinatorial types of strongly reduced maps in $[f]$.*

Proof. Let $g \in [f]$ be strongly reduced, and let \vec{e} be an oriented edge of Γ_1 . We must show that there is a finite set of possible edge-paths for $f(\vec{e})$. For this, we apply Lemma 3.33. Either f is constant on e (with only finitely many combinatorial possibilities), or else there is a multi-curve (C, c) running over e as in the statement. Since (C, c) runs over each edge of Γ_1 at most twice, there are only finitely many possibilities for it. Furthermore, $[f \circ c]$ is determined by $[f]$ and $[c]$, so there are only finitely many edge-paths in Γ_2 arising from $[f \circ c]$. The edge-path of $f(\vec{e})$ must be a sub-path of the edge-path of $f \circ c$, and again there are only finitely many possibilities. \square

Remark 3.36. Proposition 3.35 is false if we look at reduced maps rather than strongly reduced maps. (The map f can be “spun” around a taut cycle of Γ_1 .)

3.7. Restricting the domain and range.

Lemma 3.37. *Let $f: W \rightarrow \Gamma$ be a map from a marked weighted graph to a marked graph. Let $W_0 \subset W$ be the closure of the subset where $w(x) \neq 0$. Then f is taut iff the restriction $f|_{W_0}: W_0 \rightarrow \Gamma$ is taut.*

Proof. Segments of weight 0 do not effect n_f . □

Definition 3.38. A homotopy class $[f]: \Gamma_1 \rightarrow \Gamma_2$ of marked maps is *essentially surjective* if every element of $[f]$ is surjective.

Remark 3.39. If Γ_1 and Γ_2 are connected with no marked points and no unmarked 1-valent vertices, then if $[\phi]: \Gamma_1 \rightarrow \Gamma_2$ is π_1 -surjective it is essentially surjective.

If f is not (essentially) surjective, then tautness of f is equivalent to tautness of the map with restricted range.

Lemma 3.40. *Let $f: W \rightarrow \Gamma$ be a map from a marked weighted graph to a marked graph. Suppose the image of f is contained in a subgraph $\Gamma_0 \subset \Gamma$. Let Γ^* be Γ with all edges not in Γ_0 collapsed, with $\kappa: \Gamma \rightarrow \Gamma^*$ the collapsing map. Then the following are equivalent:*

- (1) f is taut,
- (2) the map with restricted range $f: W \rightarrow \Gamma_0$ is taut, and
- (3) the map $\kappa \circ f: W \rightarrow \Gamma^*$ is taut.

Proof. Recall from Theorem 3 that being taut is equivalent to being locally taut. If you replace “taut” with “locally taut” in conditions (1)–(3) above, they are easily seen to be equivalent. □

3.8. Continuity. Let $[f]: (\Gamma_1, w) \rightarrow \Gamma_2$ be a homotopy class of maps between marked graphs. We are interested in how $n_{[f]}^w$ varies as w varies. Define $N_{[f]}: \mathcal{W}(\Gamma_1) \rightarrow \mathcal{W}(\Gamma_2)$ by $N_{[f]}(w) := n_{[f]}^w$. (There is usually no single $g \in [f]$ that is taut for all w .)

Proposition 3.41. *Let $[f]: \Gamma_1 \rightarrow \Gamma_2$ be a homotopy class of maps between marked graphs. Then $N_{[f]}: \mathcal{W}(\Gamma_1) \rightarrow \mathcal{W}(\Gamma_2)$ is continuous.*

Proof. Pick $w_0 \in \mathcal{W}(\Gamma_1)$ and $e_0 \in \text{Edge}(\Gamma_2)$, and let $K = n_{[f]}^{w_0}(e_0)$. Suppose that $w \in \mathcal{W}(\Gamma_1)$ is some set of weights with $|w(e) - w_0(e)| < \varepsilon$ for all e . To show continuity of $N_{[f]}$ near w_0 , we will give upper and lower bounds on $n_{[f]}^w(e_0)$ in terms of K and ε .

To get an upper bound, let $f_0 \in [f]$ be a taut map from (Γ_1, w_0) to Γ_2 . Pick a regular value $y \in e_0$ of f_0 , and let $k = |f_0^{-1}(y)|$. Then since $f_0 \in [f]$,

$$n_{[f]}^w(e_0) \leq n_{f_0}^w(y) \leq n_{f_0}^{w_0}(y) + k\varepsilon = K + k\varepsilon,$$

as desired.

To get a lower bound, let $E_1 = \{e \in \text{Edge}(\Gamma_1) \mid w_0(e) \neq 0\}$ and $\nu = \min_{e \in E_1} w_0(e)$. Make sure $\varepsilon < \nu$, and let $M = K/(\nu - \varepsilon)$. Let $g: (\Gamma_1, w) \rightarrow \Gamma_2$ be a taut map in $[f]$. Now $n_g^w(e_0)$ can be written as an integer linear combination of weights of edges:

$$n_g^w(e_0) = \sum_{e \in \text{Edge}(\Gamma_1)} a_e w(e).$$

If $n_g^w(e_0) \geq K$, we are done. Otherwise, we must have $\sum_{e \in E_1} a_e \leq M$, and so

$$\begin{aligned} n_g^w(e_0) &\geq \sum_{e \in E_1} a_e w(e) \\ &\geq -M\varepsilon + \sum_{e \in \text{Edge}(\Gamma_1)} a_e w_0(e) \\ &\geq -M\varepsilon + K. \end{aligned}$$

The last inequality uses $n_g^{w_0}(e_0) \geq K$, which follows from the definition of K . \square

Proposition 3.42. *Let $[f]: \Gamma_1 \rightarrow \Gamma_2$ be a homotopy class of maps between marked graphs. Then $N_{[f]}: \mathcal{W}(\Gamma_1) \rightarrow \mathcal{W}(\Gamma_2)$ is piecewise-linear.*

Proof. By Proposition 3.41, we may restrict attention to positive weights $w \in \mathcal{W}^+(\Gamma_1)$. Then $n_{[f]}^w(e)$ can be computed by taking a taut representative $g \in [f]$ and finding $g^{-1}(x)$ for x a generic point on e close to one of the endpoints. The combinatorial type of g determines $g^{-1}(x)$, and Proposition 3.35 there are only finitely many possibilities for $g^{-1}(x)$. For each possibility, $n_g(e)$ is an integer linear combination of weights on Γ_1 . By definition $n_{[f]}^w(e)$ is the minimum these possibilities, so $N_{[f]}$ is a minimum of a finite set of linear functions. \square

4. WEAK GRAPHS AND LIPSCHITZ ENERGY

We now turn to minimizing the Lipschitz stretch factor of maps between graphs.

Theorem 4 (White). *Let $[\phi]: K_1 \rightarrow K_2$ be a homotopy class of maps between marked length graphs. Then there is a representative $\psi \in [\phi]$ and a marked curve $c: C \rightarrow K_1$ so that the sequence*

$$C \xrightarrow{c} K_1 \xrightarrow{\psi} K_2$$

is tight. In particular,

$$\text{Lip}[\phi] = \text{Lip}(\psi) = \frac{\ell[\psi \circ c]}{\ell[c]} = \sup_{c: C \rightarrow K_1} \frac{\ell[\psi \circ c]}{\ell[c]},$$

where the supremum runs over all non-trivial curves. We get the same quantity if we take the supremum over multi-curves or over curves that cross each edge of K_1 at most twice.

A version of Theorem 4 appears in papers by Bestvina [Bes11, Proposition 2.1] and Francaviglia-Martino [FM11, Proposition 3.11], and is attributed to Tad White (unpublished). Both papers work in the context of outer space and so assume that ϕ is a homotopy equivalence, although that assumption is never used in the proof. The extension to marked graphs is also immediate.

We need an extension of Theorem 4: we will need to understand the behavior of energies on the boundary of $\mathcal{L}^+(\Gamma)$. This leads us to *weak graphs*, in which edges may have length 0. The definition of maps between weak graphs requires a little care, since we remember homotopy information that may not be present at the level of the collapsed graph.

Definition 4.1. A *weak graph* is a graph Γ in which certain edges are designated as *null edges*; the union of the null edges forms the *null graph*. The *collapsed graph* Γ^* of Γ is the graph obtained by identifying each null edge to a single point. There is a natural *collapsing map* $\kappa: \Gamma \rightarrow \Gamma^*$. Observe that κ is a homotopy equivalence iff the null graph is a forest.

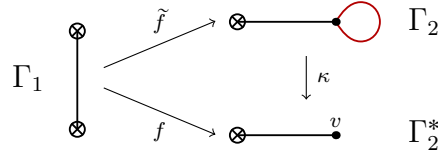


FIGURE 4. A weak map where an exact lift is impossible. On the right is a weak graph Γ_2 , with the red loop declared to be a null edge, and the collapsed graph Γ_2^* . The map $f: \Gamma_1 \rightarrow \Gamma_2^*$ maps the edge of Γ_1 forward and backward over the edge of Γ_2^* , so that the inverse image of the vertex v is a single point. The local lift $\tilde{f}: \Gamma_1 \rightarrow \Gamma_2$ maps the edge of Γ_1 around the null loop.

If (Γ, ℓ) is a weak length graph, we consider Γ to be a weak graph, where the null edges are the edges of length 0. The lengths $\ell(e)$ determine a pseudo-metric on Γ and a metric on Γ^* .

If Γ_1 and Γ_2 are weak graphs, a PL *weak map* from Γ_1 to Γ_2 is a pair (\tilde{f}, f) of a PL map $f: \Gamma_1^* \rightarrow \Gamma_2^*$ between the collapsed graphs, together with a map $\tilde{f}: \Gamma_1 \rightarrow \Gamma_2$ that is a local homotopy lift of f , in the following sense. Pick disjoint regular neighborhoods N_v^* of each vertex v of Γ_2^* . Let $N_2^* = \bigcup_v N_v^* \subset \Gamma_2^*$, $N_1^* = f^{-1}(N_2^*)$, $N_2 = \kappa_2^{-1}(N_2^*)$, and $N_1 = \kappa_1^{-1}(N_1^*)$. Then we require that $N_1 = f^{-1}(N_2)$ and that the diagram

$$(4.2) \quad \begin{array}{ccc} \Gamma_1 & \xrightarrow{\tilde{f}} & \Gamma_2 \\ \downarrow \kappa_1 & & \downarrow \kappa_2 \\ \Gamma_1^* & \xrightarrow{f} & \Gamma_2^* \end{array}$$

commutes up to a homotopy that is the identity on $\Gamma_1 \setminus N_1$.

We do not require that \tilde{f} is an exact lift of f , as those do not always exist; see Figure 4.

When we speak about an *energy* of a weak map (\tilde{f}, f) (with some additional structure on the graphs, as appropriate), we mean the energy of the map f between collapsed graphs, using the same analytic expression as for maps between ordinary graphs. This applies to all of the energies we have introduced so far, including the Dirichlet, Lipschitz, and embedding energies. On the other hand, by the *homotopy class* of the weak map we mean the homotopy class $[\tilde{f}]$ of the map between the uncollapsed graphs. If the null graphs have non-trivial π_1 , then $[f]$ will have less information than $[\tilde{f}]$.

Two weak maps (\tilde{f}, f) and (\tilde{g}, g) are equal if $f = g$ and \tilde{f} and \tilde{g} are locally homotopic.

We next generalize the notions of reduced maps.

Definition 4.3. Let $(\tilde{f}, f): (\Gamma_1, \Gamma_1^*) \rightarrow (\Gamma_2, \Gamma_2^*)$ be a weak marked map. Suppose we have point $y \in \Gamma_2^*$, a regular neighborhood N_y^* of y , and a component Z of $\kappa_1^{-1}(f^{-1}(N_y^*))$ that is not an entire component of Γ_1 . Then we say that Z is a *dead end* of f if \tilde{f} can be changed by a local homotopy so that $\tilde{f}(Z)$ does not intersect $\kappa_2^{-1}(y)$. Concretely, we have the following, where $N_y = \kappa_2^{-1}(N_y^*)$.

- (1) If Z contains a marked point, then Z is not a dead end.
- (2) If there are two points $x_1, x_2 \in \partial Z$ so that $\tilde{f}(x_1)$ and $\tilde{f}(x_2)$ are distinct points of ∂N_y , then Z is not a dead end.

- (3) If there are points $x_1, x_2 \in \partial Z$ (possibly identical) and a path γ in Z between them so that $\tilde{f}(x_1) = \tilde{f}(x_2)$ and $\tilde{f}(\gamma)$ is non-trivial in $\pi_1(N_y, \tilde{f}(x_1))$, then Z is not a dead end.
- (4) Otherwise, Z is a dead end.

If (\tilde{f}, f) has no dead ends, we say that it is *reduced*. As with non-weak maps, we can *reduce* dead ends: if Z is a dead end mapping to $y \in \Gamma_2^*$, modify f so that $f(\kappa_2(Z))$ misses y and continue to pull until you reach another singular value of f .

Proposition 4.4. *Every weak map is homotopic to a reduced map. Reduction does not increase E_q^p for any $p \leq q$.*

Proof. As in Proposition 3.4, repeatedly reduce dead ends. As in Proposition 3.5, energies are clearly not increased. \square

Theorem 4'. *Let $[\phi]: K_1 \rightarrow K_2$ be a homotopy class of maps between weak marked length graphs, so that K_1 has at least one non-null edge and so that no null cycle in K_1 maps to a non-null cycle in K_2 . Then there is a weak map $\psi \in [\phi]$ and a marked curve $c: C \rightarrow K_1$ with $\ell[c] > 0$ so that the sequence*

$$C \xrightarrow{c} K_1 \xrightarrow{\psi} K_2$$

is tight. In particular,

$$\text{Lip}[\phi] = \text{Lip}(\psi) = \frac{\ell[\psi \circ c]}{\ell[c]} = \sup_{\substack{c: C \rightarrow K_1 \\ \ell[c] > 0}} \frac{\ell[\psi \circ c]}{\ell[c]},$$

where the supremum runs over all curves c of positive length in K_1 . We get the same quantity if we take the supremum over multi-curves or over curves that cross each edge of K_1 at most twice.

To prove Theorem 4', we use a finite-dimensional approximation to the space of all maps from K_1 to K_2 . We first treat the case of ordinary (non-weak) maps.

Definition 4.5. A PL map $f: K_1 \rightarrow K_2$ between marked length graphs is *constant-derivative* if it is edge-reduced and $|f'|$ is constant on each edge of K_1 . A constant-derivative map is determined by its combinatorial type (Definition 3.34) plus a bounded amount of continuous data: for each vertex v of K_1 that maps to the interior of an edge e of K_2 , record where $f(v)$ is along e . Thus we can assemble the constant-derivative maps in $[f]$ into a locally-finite polyhedral complex $\text{PL}^*[f]$ of dimension at most $|\text{Vert}(K_1)|$. For $D > 0$ a constant, let $\text{PL}^*[f]_{\leq D}$ be the subspace of $\text{PL}^*[f]$ with Lipschitz constant $\leq D$.

Lemma 4.6. *For any homotopy class $[f]: K_1 \rightarrow K_2$ of maps between marked length graphs and any $D > 0$, the space $\text{PL}^*[f]_{\leq D}$ is a compact polyhedral complex.*

Proof. The bound on derivatives gives a bound on how many edges of K_2 the image of an edge of K_1 can cross. Therefore only finitely many different cells of $\text{PL}^*[f]$ intersect $\text{PL}^*[f]_{\leq D}$. \square

We now turn to versions of Definitions 3.34 and 4.5 for weak maps.

Definition 4.7. The *combinatorial type* of an edge-reduced weak map $(\tilde{f}, f): \Gamma_1 \rightarrow \Gamma_2$ consists of the following discrete data.

- For each vertex v of Γ_1^* , record whether $f(v)$ is on a vertex or in the interior of an edge of Γ_2^* , and which vertex or edge it lies on.
- Record the homotopy class $[\tilde{f}]$.

If we want to make this parallel to Definition 3.34, the homotopy class of \tilde{f} is conveniently recorded by picking, for each vertex v of Γ_1 , a lift of $f(\kappa_1(v))$ to a point $\tilde{f}(v) \in \Gamma_2$, and then recording, for each oriented edge \vec{e} of Γ_1 , the reduced sequence of (partial) edges that $\tilde{f}(\vec{e})$ passes over. But this is not uniquely specified by the homotopy class of \tilde{f} : different choices of lifts of $f(\kappa_1(v))$ will lead to different sequences of edges in $\tilde{f}(\vec{e})$. The combinatorial type of (\tilde{f}, f) does not depend on this auxiliary choice of lift.

Definition 4.8. A weak PL map $(\tilde{f}, f): K_1 \rightarrow K_2$ between weak marked length graphs is *constant derivative* if it is edge-reduced and $|f'|$ is constant on each non-null edge of K_1 . As before, we can assemble the constant-derivative maps in $[f]$ into a polyhedral complex $\text{PL}^*[f]$ of dimension at most $|\text{Vert}(K_1^*)|$, with one cell per combinatorial type. If we give an upper bound on $\text{Lip}(f)$, we get a subcomplex $\text{PL}^*[f]_{\leq D}$.

Lemma 4.9. *For any homotopy class $[f]: K_1 \rightarrow K_2$ of maps between weak marked length graphs and constant $D > 0$, the space $\text{PL}^*[f]_{\leq D}$ is a compact polyhedral complex.*

Proof. As in Lemma 4.6, for any non-null edge e , the bound on $\text{Lip}(f)$ gives a bound on how many edges of K_2^* can be crossed by $f(e)$. The rest of the data is the homotopy class, which is fixed by definition (and also determines the homotopy class of the local lifts). \square

Proof of Theorem 4'. We follow the previous proofs [Bes11, FM11], adapted to the setting of weak maps between marked graphs.

First, we may assume a weak map in $[\phi]$ of minimal Lipschitz energy is in $\text{PL}^*[\phi]$. The condition that no null cycle of K_1 maps to a non-null cycle of K_2 guarantees that there is some weak map $(\tilde{\psi}_0, \psi_0) \in \text{PL}^*[\phi]$ of finite Lipschitz energy. Then any optimizer $(\tilde{\psi}, \psi)$ must be in $\text{PL}^*[\phi]_{\leq \text{Lip}(\psi_0)}$. By Lemma 4.9 we are minimizing a continuous function over a compact set, so the minimum is achieved. Let $(\tilde{\psi}, \psi) \in \text{PL}^*[\phi]$ realize $\text{Lip}[\phi]$.

Next we prove the existence of a marked curve c exhibiting the Lipschitz stretch of ψ . For each non-null edge e of K_1 , say that it is

- a *tension edge* if $|\psi'(e)|$ is the maximal value, $\text{Lip}(\psi)$, and
- a *slack edge* if $|\psi'(e)| < \text{Lip}(\psi)$.

Assume that the set of tension edges of ψ is minimal among maps with minimal $\text{Lip}(\psi)$.

To find the desired curve, we will find a suitable sequence of oriented edges $(\vec{e}_i)_{i=1}^N$ of K_1 passing only over tension and null edges.

Lemma 4.10. *In the setting above, let \vec{e}_0 be an oriented tension edge of K_1 . Then we can find a reduced sequence of edges $\vec{e}_1, \dots, \vec{e}_k$ on K_1 so that either*

- *the \vec{e}_i for $i = 1, \dots, k$ are null edges and \vec{e}_k ends at a marked point, or*
- *the \vec{e}_i for $i = 1, \dots, k-1$ are null edges and \vec{e}_k is a tension edge.*

In either case, $(\psi(\vec{e}_0), \dots, \psi(\vec{e}_k))$ is also reduced.

Proof. Let C_1 be the connected component of the null graph of K_1 that contains the end of \vec{e}_0 and let $\{x\} = \kappa_1(C_1)$. Let $y = \psi(x_1)$ and C_2 be the relevant component of $\kappa_2^{-1}(y)$. C_2 may be a point on an edge of K_2 , a vertex of K_2 , or a connected null subgraph of K_2 . We proceed by cases on C_1 and C_2 , parallel to the cases in Definition 4.3.

- (1) If C_1 has a marked point x , connect the end of \vec{e}_0 to x through null edges.
- (2) If there is another tension edge \vec{f} of K_1 incident to C_1 so that \vec{f} and the reverse of \vec{e}_0 map by $\psi \circ \kappa$ to distinct directions in K_2^* , connect \vec{e}_0 to \vec{f} by a reduced path in K_1 .

- (3) If $\pi_1(C_1)$ maps non-trivially to $\pi_1(C_2)$, connect \vec{e}_0 to its reverse by a reduced loop in C_1 that maps to a non-trivial loop in C_2 .
- (4) If none of the above cases apply, we get a contradiction, as follows. Partition the non-null edges incident to C_1 according to which direction they map to from y . (Put each edge with $|\psi'| = 0$ into its own part.) Since we are not in cases (1) or (3), we can displace $y \in K_2^*$ in any direction, reducing $|\psi'|$ on one group of edges at the cost of increasing it on all other groups. Since we are not in case (2), we can reduce $|\psi'(\vec{e}_0)|$ so e_0 is no longer a tension edge without creating another tension edge, contradicting the assumption that the set of tension edges of ψ was minimal.

In any of cases (1)–(3), by construction the sequence of edges in K_1 is reduced, and the path in K_2 falls into one of the cases of Definition 4.3 and so is reduced as a weak map. \square

To complete the proof of Theorem 4', start with any tension edge \vec{e}_0 of K_1 . (There is at least one since not all edges of K_1 are null.) Use Lemma 4.10 to construct a non-backtracking chain of tension and null edges forward and backward from \vec{e}_0 . Either the chain ends in a marked point both forward and backward, or there is a repeated oriented edge. In either case we can extract a marked curve c (either an interval or a cycle) so that both c and $\psi \circ c$ are (weakly) reduced. Thus $\text{Lip}(\psi) = \ell(\psi \circ c)/\ell(c) = \ell[\psi \circ c]/\ell[c]$. If we make the null paths as efficient as possible and close off to make a cycle the first time that an oriented edge is repeated in the chain, then c will cross each (unoriented) edge of K_1 at most twice. \square

Remark 4.11. The proof of Theorem 4' is similar to the proof of Proposition 3.16. The presence of null edges introduces extra complications in Theorem 4'.

5. HARMONIC MAPS

5.1. Definition and existence. In this section, we give a concrete description of harmonic maps and prove their existence, making the intuitive description of harmonic maps from the introduction more precise. See Section 1.1 for a summary of prior work.

Definition 5.1. Let $G = (\Gamma, \alpha)$ be a marked elastic graph and let K be a marked length graph. Recall from Definition 4.5 that $\text{PL}^*[f]$ is the space of PL maps for which $|f'|$ is constant on each edge of G . To any constant-derivative map f , we can associate the *tension-weighted graph* $W_f = (\Gamma, |f'|)$; that is, the weight of an edge e is $|f'(e)|$. Then f is *harmonic* if it is constant-derivative and the associated map $f: W_f \rightarrow K$ is taut.

In general, it is not immediately obvious when a map is taut. We can simplify the condition in Definition 5.1 to give the triangle inequalities at vertices from the introduction.

Proposition 5.2. *Let $f: G \rightarrow K$ be a constant-derivative map from a marked elastic graph to a marked length graph. Let $G_0 \subset G$ be the subgraph of G on which $|f'| \neq 0$, and let f_0 be the restriction of f to G_0 . Then f is harmonic iff $(W_{f_0}, \tau(f_0))$ forms a marked weighted train track according to Definition 3.14.*

Proof. By Lemma 3.37, f is harmonic iff $f_0: W_{f_0} \rightarrow K$ is taut. If f_0 is taut, the train-track in condition (3) of Theorem 3 must be $(W_{f_0}, \tau(f_0))$. \square

Theorem 5. *Let $[f]: G \rightarrow K$ be a homotopy class of maps from a marked elastic graph to a marked length graph. Then there is a harmonic map in $[f]$. Furthermore, the following conditions are equivalent.*

- (1) *The map f is a global minimum for Dir.*

- (2) The map f is a local minimum for Dir .
- (3) The map f is harmonic.
- (4) The natural map $\iota: W_f \rightarrow G$ is part of a tight sequence

$$W_f \xrightarrow{\iota} G \xrightarrow{f} K.$$

- (5) There is a weighted multi-curve (C, c) on G that forms a tight sequence

$$C \xrightarrow{c} G \xrightarrow{f} K.$$

Proof. We start with the equivalences. By definition, (1) implies (2).

To show that (2) implies (3), let f be a local minimizer for $\text{Dir}(f)$ within $[f]$; we wish to show f is harmonic. One of the first results in calculus of variations is that f is constant-derivative. So we only need to show the triangle inequalities. Let v be a unmarked vertex of G of valence k , and let d_1, \dots, d_k be the non-zero directions of K at $f(v)$. For small $\varepsilon > 0$, let $f_{i,\varepsilon}$ be f modified by moving $f(v)$ a distance ε in the direction d_i , extended to the edges so that $f_{i,\varepsilon}$ is still constant-derivative. By hypothesis $\text{Dir}(f) \leq \text{Dir}(f_{i,\varepsilon})$. We have

$$\text{Dir}(f_{i,\varepsilon}) = \text{Dir}(f) + \sum_{d \in f^{-1}(d_i)} -2\varepsilon|f'(d)| + \sum_{j \neq i} \sum_{d \in f^{-1}(d_j)} 2\varepsilon|f'(d)| + \sum_{d \text{ direction at } v} \varepsilon^2/\alpha(d).$$

To be a local minimum, we must have for each i

$$\left. \frac{d}{d\varepsilon} (\text{Dir}(f_{i,\varepsilon})) \right|_{\varepsilon=0} \geq 0.$$

This gives the i 'th triangle inequality at v , so by Proposition 5.2, f is harmonic.

To see that (3) implies (4), suppose that f is harmonic. Then $f \circ \iota$ is taut and therefore energy-minimizing. Furthermore,

$$\begin{aligned} \text{EL}(\iota) &= \sum_{e \in \text{Edge}(\Gamma)} \alpha(e)|f'(e)|^2 = \text{Dir}(f) \\ \ell(f \circ \iota) &= \sum_{e \in \text{Edge}(\Gamma)} |f'(e)| \ell(f(e)) = \text{Dir}(f), \end{aligned}$$

so the energies are multiplicative, as desired.

Proposition 3.16 tells us that (4) implies (5).

Lemma 1.30 tells us that (4) or (5) implies (1).

Finally, to prove existence, let $g \in \text{PL}^*[f]$ be any constant-derivative map. Then $\text{Dir}[f] \leq \text{Dir}(g)$. For each edge e , this gives an upper bound on possible values of $|f'(e)|$ for any minimizer; let D be a bound for all edges. Then Dir is a continuous function on the compact complex $\text{PL}^*[f]_{\leq D}$, and therefore has a global minimum, which is harmonic. \square

5.2. Harmonic maps to weak graphs. As with Lipschitz energy, we need a generalization of Theorem 5 to allow the target to be a weak length graph.

Definition 5.3. Let $(\tilde{f}, f): \Gamma_1 \rightarrow \Gamma_2$ be weak map between marked weak graphs, and suppose there is a weight structure w_1 on Γ_1 . Then f is *taut* if there is a local lift \tilde{f}' of f so that \tilde{f}' is taut.

(Recall that in a weak map (\tilde{f}, f) , the local lift \tilde{f} is only defined up to local homotopy.)

Definition 5.4. Let $G = (\Gamma, \alpha)$ be a marked elastic graph and let K be a marked weak length graph. We say that a weak map $(\tilde{f}, f): G \rightarrow K$ is *harmonic* if it is constant-derivative and the weak map $W_f \rightarrow K$ from the tension-weighted graph is taut.

We do not allow G to be a weak graph. The definition could be extended to this case with some more work (since when e is a null edge, $|f'(e)|$ is not defined), but we will not need it.

Theorem 5'. *Let $[f]: G \rightarrow K$ be a homotopy class of maps from a marked elastic graph to a marked weak length graph. Then there is a harmonic weak map in $[f]$. Furthermore, the following conditions are equivalent.*

- (1) *The weak map f is a global minimum for Dir.*
- (2) *The weak map f is a local minimum for Dir.*
- (3) *The weak map f is harmonic.*
- (4) *There is a weighted curve (C, c) on G that forms a tight sequence*

$$C \xrightarrow{c} G \xrightarrow{f} K.$$

Proof. The most significant change from the proof of Theorem 5 is the proof that (2) implies (3), which we now do. Let $(\tilde{g}, g) \in \text{PL}^*[f]$, and suppose that g is not harmonic in the sense of Definition 5.4. Then we will find a local modification that reduces $\text{Dir}(g)$. The possible local modifications are more complicated than in the earlier proof, so we will use the technology of taut maps from Section 3.

Fix ε sufficiently small (to be specified). For v a vertex of G , let $N_v^* \subset K^*$ be the ε -neighborhood of $f(v)$. We assume that ε is small enough so that the N_v^* are disjoint regular neighborhoods. Let $N^* = \bigcup_v N_v^*$, $N_v = \kappa^{-1}(N_v^*) \subset K$, $N = \bigcup_v N_v$, $M_v = g^{-1}(N_v^*) \subset G$, and $M = \bigcup_v M_v$. We will use M and N for our neighborhoods in the definition of a weak map. On the restriction of \tilde{g} to each neighborhood, set up a marked local model, as in Definition 3.27. Choose \tilde{g} so each local model is taut. By assumption, \tilde{g} is not taut everywhere on K . The failure to be taut can only happen on ∂N . Let $y \in \partial N_v$ be one point where \tilde{g} is not locally taut, and let Z be $\tilde{g}^{-1}(y)$, minus any isolated points in the middle of edges.

The edges incident to Z can be divided into “inside” edges, those that proceed into N_v , and “outside” edges, those that proceed away from N_v . (For simplicity of notation assume that no edge has both ends incident to Z .) Since \tilde{g} is not locally taut at y , the total tension of the inside edges is not equal to the total tension of the outside edges. We must have

$$(5.5) \quad \sum_{e \text{ inside}} |g'(e)| < \sum_{e \text{ outside}} |g'(e)|,$$

as \tilde{g} restricted to M_v is taut.

Let $(\tilde{h}, h) \in \text{PL}^*[f]$ be the small modification of (\tilde{g}, g) so that h maps each vertex in Z to y and agrees with g on all other vertices. In h the length of the image of each inside edge was increased by ε , and the length of the image of each outside edge was decreased by ε . Then

$$\begin{aligned} \text{Dir}(h) - \text{Dir}(g) &= \sum_{e \text{ inside}} \frac{\ell(g(e))\varepsilon + \varepsilon^2}{\alpha(e)} + \sum_{e \text{ outside}} \frac{-\ell(g(e))\varepsilon + \varepsilon^2}{\alpha(e)} \\ &= \varepsilon \left(\sum_{e \text{ inside}} |f'(e)| - \sum_{e \text{ outside}} |f'(e)| \right) + O(\varepsilon^2), \end{aligned}$$

which is negative for ε sufficiently small by Equation (5.5). Thus $\text{Dir}(h) < \text{Dir}(g)$ and f is not a local minimum for Dir.

The rest of the proof is almost unchanged: (1) implies (2) by definition, and (4) implies (1) by properties of tight sequences. For (3) implies (4), if f is harmonic, by definition of weak harmonic maps and Theorem 3 there is a weighted multi-curve (C, c) on G so that $n_c = |f'|$. Then $C \xrightarrow{c} G \xrightarrow{f} K$ is tight. Finally, existence follows from compactness of $\text{PL}_{\leq D}^*[f]$ (Lemma 4.9). \square

We can improve the local lifts in a weak harmonic map a little.

Definition 5.6. In a weak map $(\tilde{f}, f): \Gamma_1 \rightarrow \Gamma_2$, we say that the local lift \tilde{f} is *vertex-precise* if it is strictly reduced and, for every vertex v of Γ_1 , $\kappa_2(\tilde{f}(v)) = f(\kappa_1(v))$ on the nose (with no need for homotopy).

Proposition 5.7. *If $(\tilde{f}, f): G \rightarrow K$ is a harmonic weak map, then \tilde{f} can be chosen to be vertex-precise and taut as a map from W_f to K .*

Proof. By definition of harmonic weak maps, we can find a taut initial lift \tilde{f} . If \tilde{f} does not map vertices to vertices, pick some vertex v so that $f(v) \neq \kappa(\tilde{f}(v))$. Then $\tilde{f}(v)$ lies on an edge e incident to $\kappa^{-1}(f(v))$. Since \tilde{f} is taut, $n_{\tilde{f}}$ is constant on e . We can thus push $\tilde{f}(v)$ into $\kappa^{-1}(f(v))$ without changing f or $n_{\tilde{f}}$. Repeat until \tilde{f} maps vertices to vertices. We can now straighten the edges (so that \tilde{f} is locally injective or constant on each edge). Any remaining dead ends must be inside the null sub-graphs and can be reduced away. \square

5.3. Uniqueness and continuity. Harmonic maps are not in general unique in their homotopy class. For instance, if the target length graph is a circle, then composing a harmonic map with any rotation of the circle gives another harmonic map. However, the length of the image of an edge in a harmonic map is unique. In fact, the lengths are unique in a larger set.

Definition 5.8. For Γ a marked graph, K a weak marked length graph, and $[f]: \Gamma \rightarrow K$ a homotopy class of maps, a *relaxed map* r with respect to $[f]$ is an assignment of a length $r(e)$ to each edge e of Γ , so that, for any taut weighted marked multi-curve (C, c) on Γ ,

$$(5.9) \quad \sum_{e \in \text{Edge}(\Gamma)} n_c(e) r(e) \geq \ell[f \circ c].$$

A relaxed map r naturally gives a weak length metric on Γ . Let $\text{Rlx}[f] \subset \mathcal{L}(\Gamma)$ be the set of relaxed maps. We write $\text{Rlx}_\ell[f]$ if we need to make precise the dependence on $\ell \in \mathcal{L}(K)$.

Although a relaxed map is not, in fact, any sort of map, the next three lemmas show how relaxed maps are related to actual maps.

Lemma 5.10. *If $[f]: \Gamma_1 \rightarrow \Gamma_2$ is a homotopy class, $r \in \mathcal{L}(\Gamma_1)$, and $\ell \in \mathcal{L}(\Gamma_2)$, then $r \in \text{Rlx}_\ell[f]$ iff there is a map $g \in [f]$ with $\text{Lip}_\ell^r(g) \leq 1$.*

Proof. This is Theorem 4'. \square

Lemma 5.11. *If $f: \Gamma \rightarrow K$ is a PL map, then $f^*\ell \in \text{Rlx}_\ell[f]$.*

Proof. By definition, $\text{Lip}_\ell^{f^*\ell}(f) = 1$. Apply Lemma 5.10. \square

Lemma 5.12. *If $[f]: \Gamma \rightarrow K$ is a homotopy class of maps from a marked graph Γ to a non-trivial weak marked length graph K and $r \in \text{Rlx}[f]$, then there is a PL map $g \in [f]$ so that $r = g^*\ell$.*

Proof. Lemma 5.10 gives a PL map $g_0: (\Gamma, r) \rightarrow K$ in $[f]$ with $\text{Lip}(g_0) \leq 1$. That is, $\ell(g_0(e)) \leq r(e)$ for each edge e of Γ . Define g by modifying g_0 : for each edge e on which $\ell(g_0(e)) < r(e)$, make the length of the image of e longer by introducing some zigzag folds. \square

Lemma 5.13. *Definition 5.8 does not change if we let (C, c) be*

- (1) *a marked weighted train track,*
- (2) *a marked weighted multi-curve (as in Definition 5.8),*
- (3) *a marked curve with weight 1, or*
- (4) *a marked curve with weight 1 that crosses each edge at most twice.*

Proof. The types of curve-like objects are progressively more restrictive, so we need to show that the existence of a structure of one type violating Equation (5.9) implies the next. Condition (1) implies condition (2) by Proposition 3.16. Condition (2) implies condition (3) by additivity Eq. (5.9) over connected components. Condition (3) implies condition (4) by Lemma 5.10 and Theorem 4' or, more simply, by taking any curve, looking for a maximal segment with no repeated oriented edges, doing cut-and-paste if necessary, and then using additivity over the connected components. \square

Lemma 5.14. *For $[f]: \Gamma_1 \rightarrow \Gamma_2$ a homotopy class of maps between marked graphs and $\ell \in \mathcal{L}(\Gamma_2)$, $\text{Rlx}_\ell[f]$ is a closed, non-compact, convex polytope defined by finitely many inequalities, each inequality depending linearly on ℓ .*

Proof. This follows from condition (4) of Lemma 5.13, as there are only finitely many multi-curves crossing each edge at most twice, and Equation (5.9) cuts out a closed half-space for each such multi-curve. Scaling a relaxed map by a factor $\lambda > 1$ gives another relaxed map, so $\text{Rlx}[f]$ is not compact. \square

If $r \in \text{Rlx}[f]$ is a relaxed map where the domain is an elastic graph (Γ, α) , we can define the Dirichlet energy of r :

$$(5.15) \quad \text{Dir}^\alpha(r) := \sum_{e \in \text{Edge}(\Gamma)} \frac{r(e)^2}{\alpha(e)}.$$

In fact, Equation (5.15) makes sense for any $r \in \mathcal{L}(\Gamma)$.

We can now give the uniqueness statement.

Proposition 5.16. *If $f: G \rightarrow K$ is a harmonic map from a marked elastic graph to a weak marked length graph, then $f^*\ell$ uniquely minimizes Dirichlet energy on $\text{Rlx}_\ell[f]$.*

Proof. Let $G = (\Gamma, \alpha)$. The function Dir^α is strictly convex on $\mathcal{L}(\Gamma)$, and its sub-level sets are compact. As such, Dir^α achieves a unique minimum on the closed convex set $\text{Rlx}[f]$. Since f was harmonic, it minimizes $\text{Dir}(f)$ in $[f]$; since every point in $\text{Rlx}[f]$ gives the lengths of an actual map in $[f]$, the minimizer in $\text{Rlx}[f]$ must be $f^*\ell$. \square

In light of Proposition 5.16, we can think of Dirichlet energy and the length of edges of the harmonic minimizer as functions on $\mathcal{L}(K)$.

Definition 5.17. For $[f]$ a homotopy class of maps from a marked elastic graph (Γ_1, α) to a marked graph Γ_2 , define

$$\begin{aligned} \text{Dir}_{[f]}: \mathcal{L}(\Gamma_2) &\rightarrow \mathbb{R} \\ H_{[f]}: \mathcal{L}(\Gamma_2) &\rightarrow \mathcal{L}(\Gamma_1) \end{aligned}$$

by setting $\text{Dir}_{[f]}(\ell)$ to $\text{Dir}_\ell^\alpha[f]$ and $H_{[f]}(\ell)$ to the relaxed map in $\text{Rlx}_\ell[f]$ minimizing Dir^α .

Proposition 5.18. *Let $[f]: G \rightarrow \Gamma_2$ be a homotopy class of maps from a marked elastic graph to a marked graph. Then $\text{Dir}_{[f]}$ and $H_{[f]}$ are continuous functions on $\mathcal{L}(\Gamma_2)$, with $\text{Dir}_{[f]}$ piecewise-quadratic and $H_{[f]}$ piecewise-linear.*

Proof. As $\ell \in \mathcal{L}(\Gamma_2)$ varies, the closed convex set $\text{Rlx}_\ell[f]$ varies continuously by Lemma 5.14. Since Dir^α is strictly convex, both the value and location of the minimum of Dir^α on $\text{Rlx}_\ell[f]$ depend continuously on ℓ .

Furthermore, since Dir^α is quadratic on $\mathcal{L}(\Gamma)$ and $\text{Rlx}_\ell[f]$ depends piecewise-linearly on ℓ , the value of the minimum of Dir^α on $\text{Rlx}_\ell[f]$ is a piecewise-quadratic function of ℓ and the location of the minimum is a piecewise-linear function of ℓ . (The particular quadratic or linear function depends on the face of $\text{Rlx}_\ell[f]$ containing the minimum.) \square

See also Remark 6.37.

Remark 5.19. An alternate way to see that $\text{Dir}_{[f]}$ is piecewise-quadratic and that $H_{[f]}$ is piecewise-linear is to note that they are respectively quadratic and linear for a fixed combinatorial type of a harmonic representative, and only finitely many combinatorial types of maps appear by Theorem 5' and Proposition 3.35. The combinatorial type is related to the face of $\text{Rlx}_\ell[f]$ containing $H_{[f]}(\ell)$.

6. MINIMIZING EMBEDDING ENERGY

6.1. Characterizing minimizers: λ -filling maps. We now turn to Theorem 1, starting with a characterization of which maps can appear as minimizers of $\text{Emb}[\phi]$.

Definition 6.1. A *strip graph* $S = (\Gamma, w, \ell)$ is a marked graph Γ with balanced weights $w \in \mathcal{W}(\Gamma)$ and lengths $\ell \in \mathcal{L}(\Gamma)$, so that $w(e) \neq 0$ iff $\ell(e) \neq 0$. The strip graph is *positive* if all lengths and weights are positive, and is *balanced* if w is balanced (Definition 3.14).

There is an associated marked weighted graph $W_S = (\Gamma, w)$ and weak marked length graph $K_S = (\Gamma, \ell)$. We say that S is *compatible* with an elastic graph $G_S = (\Gamma, \alpha)$ if $\ell(e) = w(e)\alpha(e)$ for each edge e . (If S is positive, then G_S is unique.) A strip graph also has an *area*

$$\text{Area}(S) := \sum_{e \in \text{Edge}(\Gamma)} \ell(e)w(e).$$

Lemma 6.2. *A balanced strip graph S gives a tight sequence of maps*

$$W_S \longrightarrow G_S \longrightarrow K_S \longrightarrow K_S^*.$$

(Recall from Definition 4.1 that K_S^* is K_S with the null edges collapsed.)

Proof. The fact that the map $W_S \rightarrow K_S$ is taut (hence energy-minimizing) is the condition that w is balanced. The map $W_S \rightarrow K_S^*$ is then taut by Lemmas 3.37 and 3.40. We also have

$$\begin{aligned} \ell(W_S \rightarrow K_S^*) &= \text{EL}(W_S \rightarrow G_S) = \text{Dir}(G_S \rightarrow K_S) = \text{Area}(S) \\ \text{Lip}(K_S \rightarrow K_S^*) &= 1, \end{aligned}$$

so the energies are multiplicative, as desired. \square

Definition 6.3. Let $S_1 = (\Gamma_1, w_1, \ell_1)$ and $S_2 = (\Gamma_2, w_2, \ell_2)$ be two marked balanced strip graphs. Write G_1 for G_{S_1} , etc. (This may involve choices.) For a PL map $\phi: \Gamma_1 \rightarrow \Gamma_2$, we also write, e.g., ϕ_G^W for the associated map from the weighted graph (Γ_1, w_1) to the elastic graph $(\Gamma_2, \ell_2/w_2)$.

Then if $\lambda > 0$ is a real number and S_2 is positive, $\phi: \Gamma_1 \rightarrow \Gamma_2$ is λ -filling if

- (1) ϕ^W is taut;
- (2) lengths are preserved: ϕ_K^K is a local isometry; and
- (3) weights are scaled by a factor of λ : for every regular value $y \in \Gamma_2$,

$$(6.4) \quad \sum_{x \in \phi^{-1}(y)} w_1(x) = \lambda w_2(y).$$

In other words, $n_\phi^{w_1} = \lambda w_2$ and in particular $\text{WR}(\phi_W^W) = \lambda$.

We also say the associated map ϕ_G^G between marked elastic graphs is λ -filling.

Lemma 6.5. *Suppose $\phi: S_1 \rightarrow S_2$ is a λ -filling map. Then $\text{Fill}_{\phi_G^G}$ is identically equal to λ . In particular, $\text{Emb}(\phi_G^G) = \lambda$.*

Proof. Since ϕ is constant on the 0-length edges of S_1 , we may assume that S_1 is positive as well. Since ϕ_K^K is length-preserving and $\alpha_i(e) = \ell_i(e)/w_i(e)$ for $i = 1, 2$ (when defined), it follows that, for regular points $x \in G_1$,

$$|(\phi_G^G)'(x)| = w_1(x)/w_2(\phi(x)).$$

The result follows from Equation (6.4). □

Lemma 6.6. *A λ -filling map $\phi: S_1 \rightarrow S_2$ gives a tight sequence of maps*

$$W_1 \longrightarrow G_1 \xrightarrow{\phi_G^G} G_2 \longrightarrow K_2.$$

Proof. The composite $W_1 \rightarrow K_2$ is taut by assumption. The energies of the various maps are

$$\begin{aligned} \text{EL}(W_1 \longrightarrow G_1) &= \text{Area}(S_1) \\ \text{Emb}(G_1 \xrightarrow{\phi_G^G} G_2) &= \lambda \\ \text{Dir}(G_2 \longrightarrow K_2) &= \text{Area}(S_2) = \text{Area}(S_1)/\lambda \\ \ell(W_1 \longrightarrow K_2) &= \text{Area}(S_1). \end{aligned}$$

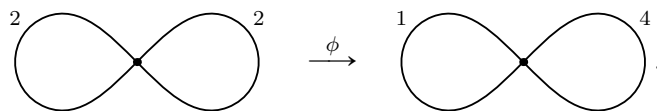
This is multiplicative, as desired. □

Proposition 6.7. *Let $\phi: G_1 \rightarrow G_2$ be a map between marked elastic graphs G_1 and G_2 . If there is a λ -filling map $\psi \in [\phi]$, then $\text{Emb}[\phi] = \text{SF}_{\text{Dir}}[\phi] = \text{SF}_{\text{EL}}[\phi] = \text{Emb}(\psi) = \lambda$.*

Proof. Immediate from Lemmas 6.5, 6.6 and 1.30. □

Thus, λ -filling maps are optimizers for Emb . If there were always a λ -filling map in $[\phi]$, then we would be done with Theorem 1. Unfortunately this is not true.

Example 6.8. Let G_1 and G_2 both be the join of two circles, with elastic constants $(2, 2)$ and $(1, 4)$, respectively, and let $\phi: G_1 \rightarrow G_2$ be the constant-derivative map which is the identity on π_1 and maps the vertex to the vertex:



Then $\text{Emb}(\phi) = 2$. Furthermore, $\text{SF}_{\text{EL}}[\phi] \geq 2$, by considering the right-hand loop of G_1 . Thus $\text{Emb}[\phi] = \text{Emb}(\phi) = \text{SF}_{\text{EL}}[\phi] = 2$. There is no λ -filling map in $[\phi]$.

Definition 6.9. Let $S_1 = (\Gamma_1, w_1, \ell_1)$ and $S_2 = (\Gamma_2, w_2, \ell_2)$ be two marked balanced strip graphs, with S_2 positive. Then, for $\lambda > 0$, a map $\phi: S_1 \rightarrow S_2$ is *partially λ -filling* if there are subgraphs $\Gamma_i^0 \subset \Gamma_i$, with induced strip structures $S_i^0 = (\Gamma_i^0, w_i, \ell_i)$, so that

- (1) $\phi(\Gamma_1^0) \subset \Gamma_2^0$;
- (2) if $\Sigma_i \subset \Gamma_i$ is the subgraph with the complementary set of edges from Γ_i^0 , then $\phi(\Sigma_1) \subset \Sigma_2$;
- (3) the restriction of ϕ to a map from S_1^0 to S_2^0 is λ -filling;
- (4) ϕ is everywhere length-preserving; and
- (5) outside of S_1^0 and S_2^0 , the map ϕ scales weights by strictly less than λ : for every regular value $y \in S_2 \setminus S_2^0$,

$$\sum_{x \in \phi^{-1}(y)} w_1(x) < \lambda w_2(y).$$

Since Γ_i^0 and Σ_i intersect at shared vertices, it is not quite always true that $\Gamma_1^0 = \phi^{-1}(\Gamma_2^0)$. We call S_1^0 and S_2^0 the *filling subgraphs* of S_1 and S_2 . There is a naturally associated marked weighted graph $W_1^0 = (S_1^0, w_1)$ and a marked length graph K_2^{0*} which is (Γ_2, ℓ_2) with all the non-filling edges collapsed to points. Note that we do not think of K_2^{0*} as a weak graph.

Lemma 6.10. *If $\phi: S_1 \rightarrow S_2$ is a partially λ -filling map then, with notation as in Definition 6.9, there is a tight sequence*

$$W_1^0 \longrightarrow G_1 \xrightarrow{\phi_G^G} G_2 \longrightarrow K_2^{0*}.$$

Proof. The version of Lemma 6.5 in this context says that $\text{Fill}_{\phi_G^G}(y) = \lambda$ if y is a regular point in S_2^0 and $\text{Fill}_{\phi_G^G}(y) < \lambda$ if y is a regular point in $S_2 \setminus S_2^0$. We then have

$$\begin{aligned} \text{EL}(W_1^0 \rightarrow G_1) &= \text{Area}(S_1^0) \\ \text{Emb}(G_1 \rightarrow G_2) &= \lambda \\ \text{Dir}(G_2 \rightarrow K_2^{0*}) &= \text{Area}(S_2^0) = \text{Area}(S_1^0)/\lambda \\ \ell[W_1^0 \rightarrow K_2^{0*}] &= \ell(W_1^0 \rightarrow K_2^{0*}) = \text{Area}(S_1^0), \end{aligned}$$

which is multiplicative, as desired. To see that the map $W_1^0 \rightarrow K_2^{0*}$ is taut, note that the map $S_1^0 \rightarrow S_2^0$ is λ -filling, which in particular implies that $W_1^0 \rightarrow \Gamma_2^0$ is taut. By Lemma 3.40 applied to $W_1^0 \rightarrow \Gamma_2$, this implies that $W_1^0 \rightarrow K_2^{0*}$ is taut. \square

The bulk of this section will be devoted to the proof of the following proposition.

Proposition 6.11. *For $[\phi]: G_1 \rightarrow G_2$ a homotopy class of maps between marked elastic graphs, there are compatible strip structures on G_1 and G_2 and map $\psi \in [\phi]$ so that ψ is partially λ -filling.*

Proposition 6.11 suffices to prove Theorem 1. We spell out some further consequences.

Proposition 6.12. *For $[\phi]: G_1 \rightarrow G_2$ a homotopy class of maps between elastic graphs, there is a tight sequence*

$$C \xrightarrow{c} T \xrightarrow{t} G_1 \xrightarrow{\psi} G_2 \xrightarrow{f} K$$

where $\psi \in [\phi]$, T is a weighted train track whose underlying graph is a subgraph of G_1 , t is the inclusion map, K is a length graph whose underlying graph is obtained by collapsing some edges of G_2 , f is the collapsing map, (C, c) is a weighted multi-curve saturating T , and $\psi \circ t$

is a train-track map. Furthermore, the edges that are not collapsed by f is the image of $\psi \circ t$ and Fill_ψ is constant and maximal on those edges.

Proof of Proposition 6.12, assuming Proposition 6.11. By Proposition 6.11, there is a partially λ -filling map $\psi \in [\phi]$. Lemma 6.10 gives a tight sequence

$$W_1^0 \xrightarrow{t} G_1 \xrightarrow{\psi} G_2 \xrightarrow{f} K_2^{0*}.$$

Set $K = K_2^{0*}$. Let T be the subgraph of W_1^0 on which $|\psi'| \neq 0$. Give T the train-track structure $\tau(\psi \circ t)$, which is well-defined by definition of T . The sequence

$$T \xrightarrow{t} G_1 \xrightarrow{\psi} G_2 \xrightarrow{f} K$$

is still tight. By Theorem 3, we can extend this to the desired 5-term tight sequence. The edges not collapsed by f , the image of $\psi \circ t$, and the edges where Fill_ψ is constant are all equal to the filling subgraph of G_2 . \square

Proof of Theorem 1, assuming Proposition 6.11. Immediate consequence of Proposition 6.12 and Lemma 1.30. \square

6.2. Iterating to optimize embedding energy. One approach to proving Proposition 6.11 would be to study those maps that locally minimize the embedding energy, analogously to the proof of Theorem 4. From a local minimum, you can extract lengths and weights to form the desired strip structure. We will take a different approach, one that also suggests an algorithm to actually compute the embedding energy.

From a homotopy class $[\phi]: G_1 \rightarrow G_2$ of maps between marked elastic graphs, we will give an explicit iteration that has a fixed point at a partially λ -filling map. To motivate the iteration, we first give an analogue in the setting of vector spaces. First, recall that a norm on a finite-dimensional vector space V defines a *dual norm* on the dual space V^* . This is the minimal norm that satisfies, for all $v \in V$ and $v^* \in V^*$,

$$(6.13) \quad \langle v^*, v \rangle \leq \|v\| \|v^*\|.$$

Equation (6.13) is tight in the sense of Proposition 2.15, namely, for every non-zero $v \in V$ there is a non-zero $v^* \in V^*$ so that Equation (6.13) is an equality. If in addition $\|v^*\| = \|v\|$, we say that v and v^* *support* each other. (The hyperplane corresponding to v^* is parallel to a supporting hyperplane at v for a norm ball in V .) We will suppose that the norms are strictly convex, which implies that supporting vectors are unique.

Example 6.14. If $\|v\| = \sqrt{\langle v, v \rangle}$ for an inner product $\langle \cdot, \cdot \rangle$, the map from a vector to its supporting vector is the canonical isomorphism $V \rightarrow V^*$ from the inner product.

Now suppose we given an isomorphism $\phi: V \rightarrow W$ between two finite-dimensional vector spaces, with a strictly convex norm on each. We wish to find the operator norm $\|\phi\|_{V,W}$ by finding a non-zero vector $v \in V$ that maximizes the ratio of norms

$$\text{NR}(v) := \frac{\|\phi(v)\|_W}{\|v\|_V}.$$

Algorithm 6.15. To attempt to maximize $\text{NR}(v)$, pick $v_0 \in V$ and set $i = 0$.

- (1) Let $w_i \in W$ be $\phi(v_i)$.
- (2) Find a supporting vector $w_i^* \in W^*$ for w_i .
- (3) Let $v_i^* \in V^*$ be $\phi^*(w_i^*)$.

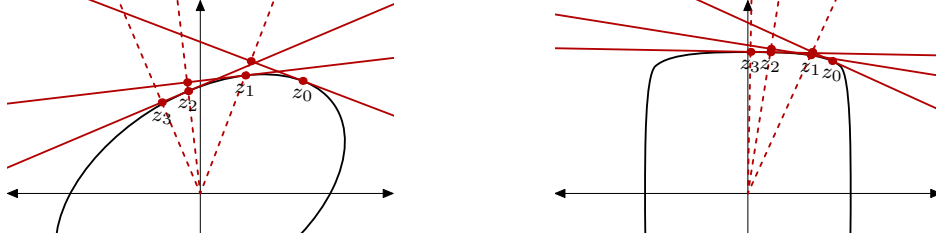


FIGURE 5. Examples for iteration on vector spaces (up to scale). On the left, B_1 comes from a quadratic norm as in Example 6.18. The right shows an example where Algorithm 6.15 does not converge to a global maximum of NR.

(4) Find a supporting vector $v_{i+1} \in V$ for v_i^* , increase i by 1, and return to Step (1).

This gives a sequence of vectors $v_i \in V$, $v_i^* \in V^*$, $w_i \in W$, and $w_i^* \in W^*$. We may also consider the corresponding sequence $[v_i]$, etc., in the respective projective spaces. The candidate for maximizing NR on PV is $\lim_{i \rightarrow \infty} [v_i]$, if it exists.

Let $\text{Iter}_\phi: V \rightarrow V$ be the composition of the steps in Algorithm 6.15, and let $P\text{Iter}_\phi: PV \rightarrow PV$ be the corresponding map on projective spaces.

Lemma 6.16. *NR(v) weakly increases under Iter_ϕ . That is, $\text{NR}(v) \leq \text{NR}(\text{Iter}_\phi(v))$. If we have equality, then v is a projective fixed point of Iter_ϕ .*

Proof. We use notation from Algorithm 6.15. Repeatedly apply Equation (6.13), as an equality for vectors that support each other:

$$\begin{aligned} \|w_i\| \|w_i^*\| &= \langle w_i^*, w_i \rangle = \langle w_i^*, \phi v_i \rangle = \langle \phi^* w_i^*, v_i \rangle = \langle v_i^*, v_i \rangle \\ \|v_i^*\| \|v_{i+1}\| &= \langle v_i^*, v_{i+1} \rangle = \langle \phi^* w_i^*, v_{i+1} \rangle = \langle w_i^*, \phi v_{i+1} \rangle = \langle w_i^*, w_{i+1} \rangle \\ \frac{\text{NR}(v_i)}{\text{NR}(v_{i+1})} &= \frac{\|w_i\| \|v_{i+1}\|}{\|v_i\| \|w_{i+1}\|} = \frac{\langle v_i^*, v_i \rangle \langle w_i^*, w_{i+1} \rangle}{\|v_i\| \|w_i^*\| \|w_{i+1}\| \|v_i^*\|} \leq \frac{\|v_i^*\| \|v_i\| \|w_i^*\| \|w_{i+1}\|}{\|v_i\| \|w_i^*\| \|w_{i+1}\| \|v_i^*\|} = 1. \end{aligned}$$

If we have equality, then $\langle v_i^*, v_i \rangle = \|v_i^*\| \|v_i\|$ and so v_i is a multiple of the supporting vector for v_i^* , namely v_{i+1} . \square

Corollary 6.17. *A vector $[v_0] \in PV$ that maximizes $\text{NR}(v_0)$ is a fixed point for $P\text{Iter}_\phi$. If v_0 is an attracting fixed point for the $P\text{Iter}_\phi$, then $\|\phi v\|/\|v\|$ has a local maximum at v_0 .*

Example 6.18. In the setting of Example 6.14, if the norms on V and W come from inner products, then Iter_ϕ is $\phi^* \phi$ and the iteration reduces to power iteration: find the maximum eigenvector of $\phi^* \phi$ by repeatedly applying it. This almost always converges to an eigenvector of maximal eigenvalue, with convergence rate determined by the ratio between the two largest distinct eigenvalues.

Example 6.19. Consider the case when ϕ is the identity on \mathbb{R}^n and $\|\cdot\|_1$ is the standard inner product. Then $\|\cdot\|_2$ is defined by its unit norm ball $B_2 \subset \mathbb{R}^n$. The supporting vector of $v \in \partial B_1$ is the tangent hyperplane to B_2 . Up to scale, Algorithm 6.15 alternates between taking a tangent hyperplane to B_1 , finding the closest point to the origin on the tangent hyperplane, and projecting to B_1 , as in Figure 5.

We now return to the actual case of interest of elastic graphs. For a marked elastic graph $G = (\Gamma, \alpha)$, we think of a “duality” between maps $f: G \rightarrow K$ to a length graph and

maps $c: W \rightarrow G$ from a weighted graph. The duality is given by $\langle c, f \rangle = \ell[f \circ c]$, and the two norms are $\sqrt{\text{Dir}[f]}$ and $\sqrt{\text{EL}[c]}$. The analogue of Equation (6.13) is Equation (2.20). More concretely, we work with $\mathcal{L}(\Gamma)$ and $\mathcal{W}(\Gamma)$, where in both cases the map (f or c) is id_Γ . The natural duality map

$$D_G: \mathcal{W}(\Gamma) \rightarrow \mathcal{L}(\Gamma)$$

$$D_G(w)(e) := \alpha(e)w(e),$$

corresponds to taking the supporting vector. Duality works best when the weights $w \in \mathcal{W}(\Gamma)$ are balanced. Fortunately this will be automatic.

Fix $[\phi]: G_1 \rightarrow G_2$ a homotopy class of maps between marked elastic graphs $G_1 = (\Gamma_1, \alpha_1)$ and $G_2 = (\Gamma_2, \alpha_2)$. For $\ell \in \mathcal{L}(\Gamma_2)$, let $\text{DR}(\ell)$ be the ratio of Dirichlet energies $\text{Dir}_\ell^{\alpha_1}[\phi]/\text{Dir}_\ell^{\alpha_2}[\text{id}_{\Gamma_2}]$, and for $w \in \mathcal{W}(\Gamma_1)$, let $\text{ER}(w)$ be the ratio of extremal lengths $\text{EL}_{\alpha_2}^w[\phi]/\text{EL}_{\alpha_1}^w[\text{id}_{\Gamma_1}]$.

Algorithm 6.20. To attempt to maximize DR and ER, pick a generic initial set of lengths $\ell_0 \in \mathcal{L}(\Gamma_2)$, and let K_0 be the marked length graph (Γ_2, ℓ_0) . We will write D_i for D_{G_i} . Let $f_0: G_2 \rightarrow K_0$ be the identity on the level of graphs. Set $i = 0$ and consider the following iteration.

- (1) Find a harmonic representative g_i of the composite map $[\phi \circ f_i]: G_1 \rightarrow K_i$. For each edge e of Γ_1 , let $m_i(e) = \ell(g_i(e))$. Thus $m_i = H_{[f]}(\ell_i)$, with $H_{[f]}$ from Definition 5.17.
- (2) Set $w_i = D_1^{-1}(m_i) \in \mathcal{W}(\Gamma_1)$, so that $w_i(e)$ is $|g'_i(e)|$, the tension weight on e . Let W_i be the weighted graph (Γ_1, w_i) , and let $d_i: W_i \rightarrow G_1$ be the identity on the level of graphs.
- (3) Set $v_i = N_{[\phi]}(w_i)$, the push-forward of w_i . Since g_i is harmonic and thus taut from the tension-weighted graph,

$$v_i(y) = \sum_{x \in g_i^{-1}(y)} w_i(x).$$

Let $c_i: W_i \rightarrow G_2$ be $g_i \circ d_i$.

- (4) Set $\ell_{i+1} = D_2(v_i) \in \mathcal{L}(\Gamma_2)$, let $K_{i+1} = (\Gamma_2, \ell_{i+1})$, and let $f_{i+1}: G_2 \rightarrow K_{i+1}$ be the identity on the level of graphs. Increase i by 1 and return to Step (1).

Schematically, Algorithm 6.20 iterates around the following loop.

$$(6.21) \quad \begin{array}{ccc} \ell_i \in \mathcal{L}(\Gamma_2) & \xleftarrow{D_2} & \mathcal{W}(\Gamma_2) \ni v_i \\ H_{[\phi]} \downarrow & & \uparrow N_{[\phi]} \\ m_i \in \mathcal{L}(\Gamma_1) & \xrightarrow{D_1^{-1}} & \mathcal{W}(\Gamma_1) \ni w_i. \end{array}$$

If all lengths and weights remain positive, we have a diagram of spaces and maps, in which

- rows are tight sequences (proved later);
- the dashed lines are only defined up to homotopy; and
- the regions commute up to homotopy.

$$(6.22) \quad \begin{array}{ccccc} W_{i-1} & \xrightarrow{c_{i-1}} & G_2 & \xrightarrow{f_i} & K_i \\ & & \nearrow [\phi] & & \parallel \\ W_i & \xrightarrow{d_i} & G_1 & \xrightarrow{g_i} & K_i \\ & & \searrow [\phi] & & \\ W_i & \xrightarrow{c_i} & G_2 & \xrightarrow{f_{i+1}} & K_{i+1} \end{array}$$

The row $W_i \xrightarrow{d_i} G_1 \xrightarrow{g_i} K_i$ is tight since g_i is harmonic and $w_i = |g'_i|$, while the row $W_i \xrightarrow{c_i} G_2 \xrightarrow{f_{i+1}} K_{i+1}$ is tight because c_i is taut and $\ell_{i+1} = D_2(v_i)$.

Let $\text{Iter}_\phi: \mathcal{L}(\Gamma_2) \rightarrow \mathcal{L}(\Gamma_2)$ be the composition

$$\text{Iter}_\phi = D_2 \circ N_{[\phi]} \circ D_1^{-1} \circ H_{[\phi]}.$$

All of the maps involved are piecewise-linear and linear on rays by Propositions 3.42 and 5.18, so Iter_ϕ is as well. We allow some of the lengths or weights to vanish (i.e., we include the boundary of $\mathcal{L}(\Gamma_i)$ and $\mathcal{W}(\Gamma_i)$ in the maps). If ϕ is essentially surjective, then $\ell \neq 0$ implies $\text{Iter}_\phi(\ell) \neq 0$, so we have a map on projective space $P \text{Iter}_\phi: P\mathcal{L}(\Gamma_2) \rightarrow P\mathcal{L}(\Gamma_2)$. Since $P\mathcal{L}(\Gamma_2)$ is a compact ball, there is a fixed point.

The iteration is parallel to the iteration on vector spaces, with

$$\begin{array}{ll} V \leftrightarrow \mathcal{L}(G_2) & W \leftrightarrow \mathcal{L}(G_1) \\ v_i \leftrightarrow f_i & w_i \leftrightarrow g_i \\ v_i^* \leftrightarrow c_i & w_i^* \leftrightarrow d_i. \end{array}$$

Remark 6.23. The only computationally-expensive step in Algorithm 6.20 is Step (1), finding the harmonic equilibrium.

Lemma 6.24. *Iter_φ increases the objective functions for both Dirichlet energy and extremal length: with the notation from Algorithm 6.20, for $i \geq 0$,*

$$\text{DR}(\ell_i) \leq \text{DR}(\ell_{i+1}) \quad \text{ER}(w_i) \leq \text{ER}(w_{i+1})$$

If we have equality in either case, then ℓ_i is a projective fixed point of Iter_φ.

Proof. This is parallel to Lemma 6.16. Tightness of the rows in Diagram (6.22) tells us

$$\begin{aligned} \text{Dir}[g_i] \text{EL}[d_i] &= \ell[g_i \circ d_i]^2 = \ell[f_i \circ \phi \circ d_i]^2 = \ell[f_i \circ c_i]^2 \\ \text{Dir}[f_{i+1}] \text{EL}[c_i] &= \ell[f_{i+1} \circ c_i]^2 = \ell[f_{i+1} \circ \phi \circ d_i]^2 = \ell[g_{i+1} \circ d_i]^2 \\ \frac{\text{DR}(\ell_i)}{\text{DR}(\ell_{i+1})} &= \frac{\text{Dir}[g_i] \text{Dir}[f_{i+1}]}{\text{Dir}[f_i] \text{Dir}[g_{i+1}]} = \frac{\ell[f_i \circ c_i]^2}{\text{Dir}[f_i] \text{EL}[d_i]} \frac{\ell[g_{i+1} \circ d_i]^2}{\text{Dir}[g_{i+1}] \text{EL}[c_i]} \\ &\leq \frac{\text{Dir}[f_i] \text{EL}[c_i] \text{Dir}[g_{i+1}] \text{EL}[d_i]}{\text{Dir}[f_i] \text{EL}[d_i] \text{Dir}[g_{i+1}] \text{EL}[c_i]} = 1. \end{aligned}$$

If we have equality, then $\ell[f_i \circ c_i]^2 = \text{Dir}[f_i] \text{EL}[c_i]$ and so $W_i \xrightarrow{c_i} G_2 \xrightarrow{f_i} K_i$ is a tight sequence, in addition to $W_i \xrightarrow{c_i} G_2 \xrightarrow{f_{i+1}} K_{i+1}$. In a tight sequence $W \xrightarrow{c} G \xrightarrow{f} K$, Equation (2.20) is an equality. The Cauchy-Schwarz inequality used in its proof then implies that $|f'|$ is proportional to n_c , which in the present case says that ℓ_i and ℓ_{i+1} must be multiples of each other, as desired for the last statement.

- $\psi(\Delta_1) \subset \Delta_2$ and $\psi(\Sigma_1) \subset \Sigma_2$; and
- the restriction of ψ to a map from (Δ_1, α_1) to (Δ_2, α_2) is λ -filling.

The map ψ satisfies the conditions to be partially λ -filling (with $\Gamma_i^0 = \Delta_i$), except for condition (5) of Definition 6.9.

Proof. Define m , w , and v from ℓ by Equation (6.27). There are tight sequences similar to Diagram (6.26), with *weak* harmonic maps (\tilde{f}, f) and (\tilde{g}, g) :

$$(6.31) \quad \begin{array}{ccccc} & & & \tilde{f} & \\ & & & \nearrow & \\ & & \Gamma_2 & \xrightarrow{\tilde{f}} & K \\ & \nearrow d & \searrow [\phi] & \searrow \tilde{g} & \downarrow \kappa \\ W & \xrightarrow{c} & \Gamma_1 & \xrightarrow{g} & K^* \\ & & \nearrow \tilde{g} & \nearrow f & \end{array}$$

Choose \tilde{g} so that $\tilde{g} \circ c$ is vertex-precise and taut, as guaranteed by Proposition 5.7.

Let $\Sigma_2 \subset \Gamma_2$ be the subgraph consisting of edges e on which v and ℓ are 0, i.e., the null subgraph of K . Let Δ_2 be the complementary subgraph.

Let $\Sigma_1 := \tilde{g}^{-1}(\Sigma_2) \subset \Gamma_1$. We first prove that Σ_1 is a subgraph of Γ_1 all of whose edges have weight 0 in w . Since $\tilde{g} \circ c$ is taut, $v = N_{[\tilde{g}]}(w)/\lambda$. Consider an edge e of Γ_1 whose interior intersects Σ_1 .

- If \tilde{g} is constant on e , then $\kappa(e)$ is a point and $m(e) = 0$, so $w(e) = 0$ as well.
- If \tilde{g} is not constant on e , consider some regular value $y \in \Sigma_2 \cap \tilde{g}(e)$ (which exists because \tilde{g} is strictly reduced). Then $v(y) = 0 = n_g^w(y)$, so $w(e) = 0$. Thus $m(e) = 0$ and $\ell(g(e)) = 0$. Since \tilde{g} is a vertex-precise lift, this implies that $\tilde{g}(e) \subset \Sigma_2$.

In either case $w(e) = 0$ and $e \subset \Sigma_1$.

Let Δ_1 be the complementary subgraph to Σ_1 . By definition of Σ_1 , we have $\tilde{g}(\Delta_1) \subset \Delta_2$ and $\tilde{g}(\Sigma_1) \subset \Sigma_2$. Since $\tilde{g}(\Sigma_1) \subset \Sigma_2$ and κ is injective on Σ_2 , we can modify the local lift \tilde{g} so that it is an exact lift of g on Σ_1 , with no homotopy required. (At this point \tilde{g} is a global exact lift of g .) Let ψ be this choice of \tilde{g} , considered as a map from G_1 to G_2 .

Let $\psi|_\Delta$ be the restriction of ψ to a map from Δ_1 to Δ_2 . We will show that

$$\psi|_\Delta: (\Delta_1, w, m) \longrightarrow (\Delta_2, v, \ell)$$

is λ -filling as a map between strip graphs. As in Proposition 6.25, the definition of m as $m(e) = \ell(g(e)) = \ell(\psi(e))$ ensures that ψ is length-preserving. We chose \tilde{g} to be taut, and by restricting first the domain to Δ_1 by Lemma 3.37 and then the range to Δ_2 by Lemma 3.40, we see that $\psi|_\Delta$ is taut. The definition of v then tells us that $\psi|_\Delta$ scales weights by λ . \square

In the setting of Proposition 6.30, let $\psi|_\Sigma$ be the restriction of ψ to a map between the collapsed subgraphs Σ_1 and Σ_2 . For $i = 1, 2$, let $P'_i = \Sigma_i \cap \Delta_i$ be the vertices shared between the two subgraphs, and let P_i be the union of P'_i with the vertices of Σ_i that were already marked. We view $\psi|_\Sigma$ as a map of marked elastic graphs $(\Sigma_1, P_1, \alpha_1) \rightarrow (\Sigma_2, P_2, \alpha_2)$, and consider the problem of finding $\text{Emb}[\psi|_\Sigma]$ in its own right.

Proposition 6.32. *In the above setting, suppose that there is a fixed point $[\ell_0]$ of $P \text{Iter}_{\psi|_\Sigma}$ with multiplier $\lambda_0 > \lambda$. Then ℓ is not a local maximum of $\text{DR}(\ell)$.*

Proof. We will show that $\text{DR}(\ell + \varepsilon \ell_0) > \text{DR}(\ell)$ for sufficiently small ε . From $\ell_0 \in \mathcal{L}(\Sigma_2)$, construct $m_0 \in \mathcal{L}(\Sigma_1)$, $w_0 \in \mathcal{W}(\Sigma_1)$, and $v_0 \in \mathcal{W}(\Sigma_2)$ by Equation (6.27). Extend ℓ_0 , m_0 , w_0 , and v_0 to Γ_i by setting them to be zero on edges in Δ_i . Let $\psi_0: (\Sigma_1, P_1) \rightarrow (\Sigma_2, P_2)$ be the

map constructed from ℓ_0 by Proposition 6.30. In particular, ψ_0 is harmonic as a map from $(\Sigma_1, P_1, \alpha_1)$ to (Σ_2, P_2, ℓ_0) . Define a new map $h: \Gamma_1 \rightarrow \Gamma_2$ by

$$(6.33) \quad h(x) := \begin{cases} \psi(x) & x \in \Delta_1 \\ \psi_0(x) & x \in \Sigma_1. \end{cases}$$

Since we pinned $\Delta_i \cap \Sigma_i$ in $\psi|_{\Sigma}$, the map h is continuous. By construction, h is weakly harmonic as a map from (Γ_1, α_1) to (Γ_2, ℓ) . We claim that h is also harmonic if we perturb ℓ . For small ε , consider the modified lengths $\ell_\varepsilon = \ell + \varepsilon \ell_0 \in \mathcal{L}(\Gamma_2)$. Let h_ε be h considered as a map from (Γ_1, α_1) to $(\Gamma_2, \ell_\varepsilon)$.

Claim 6.34. For ε sufficiently small, h_ε is a harmonic map.

Proof. For simplicity, we suppose that ℓ_0 is non-zero on every edge, so that $(\Gamma_2, \ell_\varepsilon)$ is a length graph and h_ε is an ordinary map (not weak). The case when ℓ_0 has some zeroes can be treated by induction.

The tension weight of h is $w \in \mathcal{W}(\Gamma_1)$. Let w_ε be the tension weight of h_ε . Concretely,

$$w_\varepsilon(e) = \begin{cases} w(e) & e \in \text{Edge}(\Delta_1) \\ \varepsilon w_0(e) & e \in \text{Edge}(\Sigma_1). \end{cases}$$

We must check that h_ε is still taut as a map from $(\Gamma_1, w_\varepsilon)$ to Γ_2 . By Proposition 5.2, we must check that w_ε satisfies the train-track triangle inequalities at the vertices of Γ_1 .

For vertices in $\Delta_1 \setminus \Sigma_1$ and $\Sigma_1 \setminus \Delta_1$, the triangle inequalities follow from the fact that ψ and ψ_0 are harmonic, respectively. For vertices in $\Delta_1 \cap \Sigma_1$, the triangle inequalities follow from Lemma 6.35 below, where the a_i are the weights of the incident groups of edges of Δ_1 and the b_i are the weights of the incident groups of edges of Σ_1 . \square

Lemma 6.35. If $(a_1, \dots, a_n) \in \mathbb{R}_+^n$ satisfies the triangle inequalities and $(b_1, \dots, b_m) \in \mathbb{R}_+^m$ is another vector, then, for all ε sufficiently small,

$$(a_1, \dots, a_n, \varepsilon b_1, \dots, \varepsilon b_m)$$

satisfies the triangle inequalities.

Proof. Elementary. \square

Returning to the proof of Proposition 6.32, since h_ε is harmonic, if f and g are the harmonic maps to (Γ_2, ℓ) and f_0 and g_0 are the harmonic maps to (Σ_2, ℓ_0) , we have

$$\text{DR}(\ell_\varepsilon) = \frac{\text{Dir}(h_\varepsilon)}{\text{Dir}(f) + \varepsilon^2 \text{Dir}(f_0)} = \frac{\text{Dir}(g) + \varepsilon^2 \text{Dir}(g_0)}{\text{Dir}(f) + \varepsilon^2 \text{Dir}(f_0)} > \frac{\text{Dir}(g)}{\text{Dir}(f)} = \lambda = \text{DR}(\ell)$$

using the assumption that $\lambda_0 = \text{Dir}(g_0)/\text{Dir}(f_0) > \lambda$. \square

We can now prove the existence of a partially λ -filling map.

Proof of Proposition 6.11. First, if $[\phi]$ is not essentially surjective, we can restrict to the essential image of $[\phi]$, the image of a taut representative of $[\phi]$ with respect to any weight structure on G_1 . After doing this, $P \text{Iter}_\phi$ is defined.

We proceed by induction on the size of Γ_1 . Given the homotopy class $[\phi]$, let $\ell \in \mathcal{L}(\Gamma_1)$ be a global maximum of DR (which exists since $P\mathcal{L}(\Gamma_1)$ is compact). By Lemma 6.24, ℓ is a fixed point of $P \text{Iter}_\phi$; let λ be its multiplier. If ℓ is in $\mathcal{L}^+(\Gamma_1)$, we are done by Proposition 6.25. Otherwise, consider the subgraphs Δ_i and Σ_i given by Proposition 6.30, with restricted maps

$\psi|_\Delta$ (which is λ -filling) and $\phi|_\Sigma$. Since Σ_1 is a proper subgraph of Γ_1 , by induction we can find a partially λ_1 -filling map $\psi|_\Sigma \in [\phi_\Sigma]$ for some $\lambda_1 \geq 0$.

Now assemble $\psi|_\Delta$ and $\psi|_\Sigma$ to a single map $\psi: \Gamma_1 \rightarrow \Gamma_2$ by Equation (6.33). Then ψ is λ -filling on Δ_1 and partially λ_1 -filling on Σ_1 . By Proposition 6.32, $\lambda_1 \leq \lambda$, so ψ is partially λ -filling on all of Γ_1 . \square

Remark 6.36. The map constructed above has a stronger “layered” structure, where Γ_1 and Γ_2 are divided into layers, each with its own filling constant. Specifically, there are properly nested subgraphs

$$\begin{aligned}\Gamma_1 &= \Sigma_1^0 \supsetneq \Sigma_1^1 \supsetneq \cdots \supsetneq \Sigma_1^n = \emptyset \\ \Gamma_2 &= \Sigma_2^0 \supsetneq \Sigma_2^1 \supsetneq \cdots \supsetneq \Sigma_2^n,\end{aligned}$$

a map $\psi: \Gamma_1 \rightarrow \Gamma_2$ in $[\phi]$, and a sequence of filling constants $\lambda_0 > \lambda_1 > \cdots > \lambda_n = 0$, with the following properties.

- (1) ψ preserves Σ^i : for $1 \leq i \leq n$, $\psi(\Sigma_1^i) \subset \Sigma_2^i$.
- (2) ψ preserves $\Sigma^i \setminus \Sigma^{i+1}$: for $k \in \{1, 2\}$ and $0 \leq i \leq n-1$, let Δ_k^i be the graph whose edges are in $\Sigma_k^i \setminus \Sigma_k^{i+1}$. Then $\psi(\Delta_1^i) \subset \Delta_2^i$.
- (3) ψ is λ_i -filling on Δ^i : for $0 \leq i \leq n$, let ψ_i be the restriction of ψ to a map from Δ_1^i to Δ_2^i , where for $i > 0$ we additionally mark the vertices in $\Delta_1^i \cap \Delta_1^{i-1}$ and in $\Delta_2^i \cap \Delta_2^{i-1}$. Then ψ_i is λ_i -filling.

Remark 6.37. The maps in Iter_ϕ can be given an interpretation in terms of derivatives. Specifically, let $\text{Dir}_{[\phi]}: \mathcal{L}(\Gamma_2) \rightarrow \mathbb{R}$ be Dirichlet energy as a function of lengths from Proposition 5.18. Then it is C^1 with derivative given by

$$d\text{Dir}_{[\phi]}(\ell) = 2 \cdot N_{[\phi]}(D_{G_1}^{-1}(H_{[\phi]}(\ell))) \in \mathcal{W}(\Gamma_2) \subset \mathcal{L}(\Gamma_2)^*$$

where we identify $\mathcal{W}(\Gamma_2)$ with a subset of $\mathcal{L}(\Gamma_2)^*$ using the duality pairing. Recall from the introduction that, physically, tension in a spring is the derivative of energy as the length varies. Thus $d\text{Dir}_{[\phi]}(\ell)$ gives the total tension in each edge of Γ_2 .

6.4. General targets. We turn to Theorem 2, allowing more general length space targets. First we need to generalize Equation (2.23) to this setting.

Lemma 6.38. *Let G_1 and G_2 be elastic graphs, let X be a length space, let $\phi: G_1 \rightarrow G_2$ be a piecewise-linear map, and let $f: G_2 \rightarrow X$ be a Lipschitz map. Then*

$$\text{Dir}(f \circ \phi) \leq \text{Emb}(\phi) \text{Dir}(f).$$

Proof. We compute

$$\begin{aligned}\text{Dir}(f \circ \phi) &= \int_{x \in G_1} |(f \circ \phi)'(x)|^2 dx \\ &= \int_{y \in G_2} \left(\sum_{x \in \phi^{-1}(y)} |\phi'(x)| \right) |f'(y)|^2 dy \\ &\leq \text{Emb}(\phi) \text{Dir}(f),\end{aligned}$$

In the second line we do a change of variables from G_1 to G_2 , using $dx = |\phi'(x)| dy$. (Any portions of G_1 where ϕ is constant and so $\phi^{-1}(y)$ is uncountable do not contribute to the integrals.) In the last line we use $\int |a| \cdot |b| dy \leq \text{ess sup } |a| \cdot \int |b| dy$. \square

Proof of Theorem 2. Suppose $\text{Emb}(\phi)$ is minimal within the homotopy class $[\phi]$ and that $\text{Dir}(f)$ is within a multiplicative factor of ε of the infimum. Then

$$\text{Dir}[f \circ \phi] \leq \text{Dir}(f \circ \phi) \leq \text{Emb}(\phi) \text{Dir}(f) \leq \text{Emb}[\phi] \text{Dir}[f](1 + \varepsilon).$$

Since we can choose ε as small as we like, this gives one inequality of the desired equality. The other direction comes from Theorem 1. \square

6.5. Algorithmic questions. Given a homotopy class $[\phi]: G_1 \rightarrow G_2$ of maps between marked elastic graphs, we have proved that Iter_ϕ has a projectively fixed set of lengths $\ell \in \mathcal{L}(G_2)$ maximizing $\text{DR}(\ell)$ and giving a partially λ -filling map in $[\phi]$. The lengths maximizing $\text{DR}(\ell)$ need not be unique.

Example 6.39. Let G_1 and G_2 both be the join of two circles, as in Example 6.8, with all elastic weights equal to 1, and let ϕ be the identity map. Then Iter_ϕ is the identity and DR is constant on $\mathcal{L}(G_2)$.

Question 6.40. What is the structure of the subset of $\mathcal{L}(G_2)$ on which DR reaches its maximum value? For instance, is it a convex subset of $\mathcal{L}(G_2)$? Can it be a proper subset of the interior of $\mathcal{L}(G_2)$?

As for convergence, we would like to say that Algorithm 6.20 works, in the sense that iterating $P\text{Iter}_\phi$ always converges to a maximum of DR (which also computes $\text{Emb}[\phi]$). The presence of extra fixed points of Iter_ϕ means that this does not always happen. (In dynamical terms, DR is only weakly increased by Iter_ϕ , not strictly increased; so DR is not quite a Lyapunov function for this discrete dynamical system.) But we can make some statements.

Proposition 6.41. *Algorithm 6.20 gives a sequence of lengths $\ell_i \in \mathcal{L}(G_2)$ that converge projectively to a fixed point for $P\text{Iter}_\phi$.*

Proof. By Lemma 6.24, $\text{DR}(\ell_i)$ weakly increases, with an upper bound; thus $\text{DR}(\ell_i)$ has a limit, and the $[\ell_i]$ have an accumulation point $[\ell_\infty]$ with $\text{DR}(\text{Iter}_\phi(\ell_\infty)) = \text{DR}(\ell_\infty)$. By Lemma 6.24 again, $[\ell_\infty]$ is a fixed point of $P\text{Iter}_\phi$, and therefore the $[\ell_i]$ limit to $[\ell_\infty]$, without the need to pass to a subsequence. \square

Lemma 6.42. *For $[\phi]: G_1 \rightarrow G_2$ a homotopy class of maps between marked elastic graphs, the set $\{\text{DR}(\ell) \mid \ell \text{ a fixed point of } P\text{Iter}_\phi\}$ is finite.*

Proof. Let $\ell \in \mathcal{L}(G_2)$ be a projective fixed point for Iter_ϕ with multiplier λ . Proposition 6.30 gives a λ -filling map on subgraphs $\phi|_\Delta: \Delta_1 \rightarrow \Delta_2$. By Proposition 6.7, $\lambda = \text{Emb}[\phi|_\Delta]$ and thus depends only on the subgraphs Δ_i . Since there are only finitely many subgraphs, we are done. \square

Proposition 6.43. *For $\phi: G_1 \rightarrow G_2$ a homotopy class of maps between marked elastic graphs, there is an open subset of $P\mathcal{L}(G_2)$ on which $P\text{Iter}_\phi$ converges to a maximum of DR .*

Proof. Let λ be the maximum value of DR on $P\mathcal{L}(G_2)$. By Lemma 6.42, there is an $\varepsilon > 0$ so that there are no fixed points of $P\text{Iter}_\phi$ in $\text{DR}^{-1}(\lambda - \varepsilon, \lambda)$. Then by Proposition 6.41, $P\text{Iter}_\phi$ converges to a maximum of DR on $\text{DR}^{-1}(\lambda - \varepsilon, \lambda]$. \square

Question 6.44. Does Algorithm 6.20 converge to a maximum of DR for an open dense set of initial data?

A few words are in order on why Question 6.44 is not as easy as it may appear. If $[\ell_1]$ is a fixed point of $P\text{Iter}_\phi$ with $\text{DR}(\ell_1) < \text{Emb}[\phi]$, then by Proposition 6.32 it is not a local maximum of DR. If Iter_ϕ were linear, that would imply the set attracted to $[\ell_1]$ has empty interior. Since Iter_ϕ is only PL, the situation is more complicated. For instance, Iter_ϕ can map an open subset of $\mathcal{L}^+(G_2)$ to a subset $S \subset \partial\mathcal{L}(G_2)$, since harmonic maps can generically map vertices to vertices. Then S could potentially be attracted to $[\ell_1]$.

We can nevertheless improve Algorithm 6.20 to always find $\text{Emb}[\phi]$.

Algorithm 6.45. Given an essentially surjective homotopy class $[\phi]: G_1 \rightarrow G_2$ of maps between marked elastic graphs, to find $\text{Emb}[\phi]$, pick arbitrary non-zero initial lengths $\ell_0 \in \mathcal{L}(G_2)$ and iterate Algorithm 6.20 to get a sequence of lengths ℓ_i . Since Iter_ϕ is piecewise-linear, each set of lengths ℓ_i is in a closed cone of linearity $D_i \subset \mathcal{L}(G_2)$. Since there are only finitely many domains of linearity, there must be some i and k so that $D_{i+k} = D_i$. By standard linear algebra techniques, we can see if $(\text{Iter}_\phi)^k$ has a projective fixed point in D_i . By Proposition 6.41, the ℓ_i converge to a projective fixed point, so eventually the linear algebra check will succeed, giving projectively fixed lengths ℓ_∞ with multiplier λ_∞ .

If ℓ_∞ is non-zero on every edge, we are done by Proposition 6.25. Otherwise, apply Proposition 6.30 to extract a map

$$\psi|_\Sigma: (\Sigma_1, P_1, \alpha_1) \rightarrow (\Sigma_2, P_2, \alpha_2)$$

between simpler graphs, with its own embedding energy λ_Σ (which we can find by induction). If $\lambda_\Sigma < \lambda_\infty$, we have found a partially λ_∞ -filling map and are done. Otherwise, by Proposition 6.32, we can find nearby lengths ℓ'_0 with $\text{DR}(\ell'_0) > \text{DR}(\ell_\infty) \geq \text{DR}(\ell_0)$. In this case, repeat the algorithm, with ℓ'_0 in place of ℓ_0 . By Lemma 6.42, eventually we will find the true maximum value of $\text{DR}(\ell)$, and thus the true value of $\text{Emb}[\phi]$.

In practice, Algorithm 6.45 appears to run quickly, at least for small examples. The additional steps to continue past a projective fixed point have been unnecessary. Theoretically, there is no reason to expect it to always perform well. In particular, in the closely related case of pseudo-Anosov maps, Bell and Schleimer have given examples where the analogous algorithm is slow [BS15].

APPENDIX A. GENERAL GRAPH ENERGIES

As suggested by the notation in Definition 1.26, the energies of this paper fit into a more general framework. We start with a notion of p -conformal graphs, simultaneously generalizing weighted graphs, elastic graphs, and length graphs. There are several different perspectives. A p -conformal graph can be viewed as

- a graph with a p -length $\alpha(e)$ on each edge;
- an equivalence class of strip graphs (Γ, w, ℓ) under a rescaling operation; or
- an equivalence class of spaces X with a length metric ℓ and measure μ under another rescaling operation.

We start with the metric view, since it is most standard, although the formulas may appear unmotivated.

Definition A.1. For $p \in (1, \infty]$, a p -conformal graph $G^p = (\Gamma, \alpha)$ is a graph with a positive p -length $\alpha(e)$ on each edge e , which gives a metric. For $p = 1$, a 1-conformal graph is a weighted graph. We will sometimes think about a metric graph as a 1-conformal graph (implicitly with weight 1), but in this case the energies are independent of the metric.

Definition A.2. For $f: G^p \rightarrow K$ a PL map from a p -conformal graph to a length graph, define

$$(A.3) \quad E^p(f) := \|f'\|_{p,G}.$$

(We take the L^p norm of $|f'|$ with respect integration with respect to α , and use an additional subscript to make clear where the norm is being evaluated.) If furthermore f is constant-derivative and $1 < p < \infty$, then

$$E^p(f) = \left(\sum_{e \in \text{Edge}(G)} \frac{\ell(f(e))^p}{\alpha(e)^{p-1}} \right)^{1/p}.$$

For $f: W \rightarrow G^p$ a PL map from a weighted graph to a p -conformal graph, define

$$(A.4) \quad E_p(f) := \|n_f\|_{p^\vee, G},$$

where $p^\vee = p/(p-1)$ is the Hölder conjugate of p . In general, for $f: G^p \rightarrow H^q$ a PL map from a p -conformal graph to a q -conformal graph with $1 \leq p \leq q \leq \infty$ and $p < \infty$, define

$$(A.5) \quad \text{Fill}^p(f): H^q \rightarrow \mathbb{R}_{\geq 0}$$

$$(A.6) \quad \text{Fill}^p(f)(y) := \sum_{x \in f^{-1}(y)} |f'(x)|^{p-1}$$

$$(A.7) \quad E_q^p(f) := (\|\text{Fill}^p(f)\|_{q/q-p, H})^{1/p}.$$

If $p < q$, this is

$$(A.8) \quad E_q^p(f) = \left(\int_H \text{Fill}^p(f)(y)^{\frac{1}{1-p/q}} d\alpha(y) \right)^{1/p-1/q}.$$

Energies of homotopy classes are defined as an infimum as usual:

$$E_p[f] := \inf_{g \in [f]} E_p(g) \quad E_q^p[f] := \inf_{g \in [f]} E_q^p(g) \quad E^p[f] := \inf_{g \in [f]} E^p(g).$$

Remark A.9. As with the earlier energies, in each case $E_q^p(f)$ naturally extends to a wider class of graph maps than PL maps, with the precise class of maps depending on p and q .

Proposition A.10. For PL maps f as above, $E_p(f) = E_p^1(f)$ and, if $p < \infty$, $E^p(f) = E_\infty^p(f)$.

Proof sketch. Change of variables. □

In light of Proposition A.10, define $E_\infty^\infty(f) := E^\infty(f) = \text{Lip}(f)$.

Proposition A.11. For $1 \leq p \leq q \leq \infty$ and $\phi: G^p \rightarrow H^q$ a PL map,

$$E_q^p(\phi) = \sup_{f: H \rightarrow K} \frac{E^p(f \circ \phi)}{E^q(f)}$$

where the supremum runs over all metric graphs K and PL maps $f: H^q \rightarrow K$ with non-zero energy. If $p < q < \infty$, equality holds exactly when $|f'|$ is proportional to $(\text{Fill}^p(\phi))^{1/(q-p)}$.

Observe that Proposition A.11 is about energies of concrete maps, not about homotopy classes of maps.

Proof. If $q = \infty$, this is easy. Otherwise, we may as well assume that K has the same underlying graph as H (with varying metric) and f is the identity as a graph map. Then we are essentially picking $|f'|$ as a piecewise-constant function on H . We have

$$\begin{aligned} E^p(f \circ \phi) &= \left(\int_G |f'(\phi(x))|^p |\phi'(x)|^p dx \right)^{1/p} \\ &= \left(\int_H |f'(y)|^p (\text{Fill}^p(\phi)(y)) dy \right)^{1/p} \\ &\leq \|f'\|_q (\|\text{Fill}^p(\phi)\|_{q/(q-p)})^{1/p} \\ &= E^q(f) E_q^p(\phi), \end{aligned}$$

using Hölder's inequality in the form $\|ab\|_1 \leq \|a\|_{q/p} \|b\|_{q/(q-p)}$. The equality statement follows from the equality condition in Hölder's inequality. \square

Remark A.12. It is also true that

$$E_q^p(\phi) = \sup_{c: W \rightarrow G} \frac{E_q(\phi \circ c)}{E_p(c)}$$

where the supremum runs over all weighted graphs (W, c) on G with non-zero energy (not necessarily taut). Equality holds when n_c is proportional to $|\phi'|^{p-1} (\text{Fill}^p(\phi))^{(p-1)/(q-p)}$.

Proposition A.13. *Given $1 \leq p \leq q \leq r \leq \infty$ and a sequence $G_1^p \xrightarrow{f} G_2^q \xrightarrow{g} G_3^r$ of maps between a marked p -conformal graph G_1 , a marked q -conformal graph G_2 , and a marked r -conformal graph G_3 ,*

$$\begin{aligned} E_r^p(g \circ f) &\leq E_q^p(f) E_r^q(g) \\ E_r^p[g \circ f] &\leq E_q^p[f] E_r^q[g]. \end{aligned}$$

Proof sketch. The first equation is an immediate consequence of Proposition A.11. The second equation follows as in the proof of Proposition 2.15. \square

Taut maps automatically minimize E_p within their homotopy class, as in Proposition 3.8. We can also minimize E^p within a homotopy class.

Proposition A.14. *Let $[f]: G^p \rightarrow K$ be a homotopy class of maps from a marked p -conformal graph to a marked length graph. Then there is a constant-derivative PL map $g \in [f]$ so that $E^p(g) = E^p[f]$.*

Proof sketch. For $p = \infty$, this is Theorem 4. For $p < \infty$, the form of E^p guarantees that an energy minimizer will be constant-derivative. As in Theorem 5, this reduces the space of possibilities to the compact polyhedron $\text{PL}^*[f]_{\leq D}$ for suitable D , which in turn guarantees there will be a minimizer. \square

A map that minimizes $E^p(f)$ within $[f]$ is called a *p -harmonic map*. Theorem 5' may also be extended to give a local characterization of p -harmonic maps, including cases where the target is weak. Specifically, $[f]: G^p \rightarrow K$ is p -harmonic iff the map $W_f \rightarrow K$ is taut, where W_f is G with p -tension weights $w(x) = |f'(x)|^{p-1}$.

There are also versions of stretch factors: for $[\phi]: G^p \rightarrow H^q$ a homotopy class of marked maps, define

$$(A.15) \quad \overrightarrow{\text{SF}}_q^p[\phi] := \sup_{[f]: H \rightarrow K} \frac{E^p[f \circ \phi]}{E^q[f]}$$

$$(A.16) \quad \overleftarrow{\text{SF}}_q^p[\phi] := \sup_{[c]: W \rightarrow G} \frac{E_q[\phi \circ c]}{E_p[c]},$$

where in (A.15) we maximize over marked length graphs K and PL maps $f: H^q \rightarrow K$ and in (A.16) we maximize over all marked weighted graphs (or multi-curves) on G^p .

Theorem 6. *For $1 \leq p \leq q \leq \infty$ and $[\phi]: G^p \rightarrow H^q$ a homotopy class of maps from a marked p -conformal graph to a marked q -conformal graph, there is a map $\psi \in [\phi]$, a marked weighted graph W , a marked weak length graph K , and a tight sequence of marked maps*

$$W \xrightarrow{c} G^p \xrightarrow{\psi} H^q \xrightarrow{f} K.$$

In particular,

$$E_q^p(\psi) = E_q^p[\phi] = \frac{E^p(f \circ \psi)}{E^q(f)} = \overrightarrow{\text{SF}}_q^p[\phi] = \frac{E_q(\psi \circ c)}{E_p(c)} = \overleftarrow{\text{SF}}_q^p[\phi].$$

Theorem 6 is analogous to Theorem 1 (and is much harder than Proposition A.11).

Proof sketch. The proof is quite similar the proof of Theorem 1 in Section 6. For $p = 1$, the tautness results of Section 3 give the result, while for $q = \infty$ this is Proposition A.14. So assume that $1 < p \leq q < \infty$.

For G^p a p -conformal graph with $1 < p < \infty$, there is an invertible “duality” map $D_G^p: \mathcal{W}(G) \rightarrow \mathcal{L}(G)$, defined by setting

$$(A.17) \quad (D_G^p(w))(e) = \alpha(e)w(e)^{1/(p-1)}.$$

Then, for a homotopy class as in the statement of Theorem 6, define an iteration $\text{Iter}_\phi: \mathcal{L}(\Gamma_2) \rightarrow \mathcal{L}(\Gamma_2)$ as follows.

- (1) For $\ell \in \mathcal{L}(\Gamma_2)$, set $K = (\Gamma_2, \ell)$, find a p -harmonic representative g of $[\text{id} \circ \phi]: G^p \rightarrow K$, and set $m(e) = \ell(g(e)) \in \mathcal{L}(\Gamma_1)$.
- (2) Set $w = (D_1^p)^{-1}(m) \in \mathcal{W}(\Gamma_1)$, so (Γ_1, w) is the tension-weighted graph of g .
- (3) Set $v = N_{[\phi]}(w) \in \mathcal{W}(\Gamma_2)$.
- (4) Set $\text{Iter}_\phi(\ell) = D_2^p(v) \in \mathcal{L}(\Gamma_2)$.

If $p = q$, Iter_ϕ descends to a map on projective spaces $P\text{Iter}_\phi: P\mathcal{L}(\Gamma_2) \rightarrow P\mathcal{L}(\Gamma_2)$, and one has to do an analysis of the possible boundary fixed points, entirely parallel to Section 6.3.

If $p < q$, then Iter_ϕ is not linear on rays, and we do the iteration on $\mathcal{L}(\Gamma_2)$ itself. More specifically, on a ray we have

$$\text{Iter}_\phi(\lambda\ell) = \lambda^{\frac{p-1}{q-1}} \text{Iter}_\phi(\ell).$$

Since $(p-1)/(q-1) < 1$, if a ray in $\mathcal{L}(\Gamma_2)$ is mapped to itself then there is a finite fixed point on the ray. A little more analysis shows that there are never attracting fixed points on the boundary, so there must be a fixed point in the interior. This gives strip graphs compatible with G^p and H^q in the sense of Definition A.18 below, along with a 1-filling map between them. \square

We now turn to the first alternate definition of p -conformal graphs and E_q^p .

Definition A.18. For $p \in [1, \infty)$, a p -conformal rescaling of a positive strip graph (Γ, w, ℓ) changes the weight, length, and area by

$$(w, \ell, \text{Area}) \mapsto (\lambda^{p-1}w, \lambda\ell, \lambda^p \text{Area})$$

where $\lambda: \text{Edge}(\Gamma) \rightarrow \mathbb{R}_+$ is a positive rescaling factor on each edge. (The identity $\text{Area} = w \cdot \ell$ is preserved.) An ∞ -conformal rescaling instead rescales acts by $(w, \ell, \text{Area}) \mapsto (\lambda w, \ell, \lambda \text{Area})$. We write $(\Gamma, w_1, \ell_1) \equiv_p (\Gamma, w_2, \ell_2)$ if the two strip structures are related by a p -conformal rescaling.

For $p \in (1, \infty]$, we say that a p -conformal graph (Γ, α) is *compatible* with a positive strip structure (Γ, w, ℓ) if

$$(\Gamma, w, \ell) \equiv_p (\Gamma, 1, \alpha),$$

or equivalently if $\ell(e) = \alpha(e)w(e)^{1/(p-1)}$. Thus we may think of a p -conformal graph as an equivalence class of \equiv_p . We say that (Γ, α) is compatible with an arbitrary (not necessarily positive) strip structure if, for each edge e ,

$$\ell(e)^{p-1} = \alpha(e)^{p-1}w(e).$$

A 1-conformal graph is compatible with a strip structure if the weights agree.

For $p = 1, 2$, or ∞ , a p -conformal graph is the same as a weighted graph, an elastic graph, or a length graph, respectively. The duality map D^p from Equation (A.17) is natural from this definition. Suppose we have a p -conformal graph (Γ, α) . Then for $w \in \mathcal{W}^+(G)$, the lengths $D^p(w)$ are the unique values so that $(\Gamma, w, D^p(w)) \equiv_p (\Gamma, 1, \alpha)$.

Definition A.19. Let $S_1 = (\Gamma_1, w_1, \ell_1)$ and $S_2 = (\Gamma_2, w_2, \ell_2)$ be two marked strip graphs (not necessarily balanced), with S_2 positive. For $\lambda > 0$, a map $f: S_1 \rightarrow S_2$ is *weakly λ -filling* if it satisfies Conditions (2) and 3 of Definition 6.3, dropping the condition that f be taut as a map between weight graphs.

Definition A.20. For $1 \leq p \leq q \leq \infty$, let G_1^p be a p -conformal graph, G_2^q be a q -conformal graph, and $f: G_1^p \rightarrow G_2^q$ be a PL map. Then $E_{q,\text{strip}}^p(f)$ is defined in the following way.

- (1) If $p < q$, there are unique strip graphs S_1 and S_2 , compatible with subdivisions of G_1^p and G_2^q respectively, so that f is weakly 1-filling as a map from S_1 to S_2 . Then

$$(A.21) \quad E_{q,\text{strip}}^p(f) = \text{Area}(S_2)^{1/p-1/q}.$$

- (2) If $p = q < \infty$, there are in general no strip structures as above. Instead, take any strip structures (Γ_1, w_1, ℓ_1) and (Γ_2, w_2, ℓ_2) as above so that f is length-preserving, and define the energy to be the maximum ratio of weights:

$$(A.22) \quad E_{p,\text{strip}}^p(f) = \text{ess sup}_{y \in \Gamma_2} \frac{n_f^{w_1}(y)}{w_2(y)}.$$

This is independent of the choices of strip structures.

Proposition A.23. For $\phi: G_1^p \rightarrow G_2^q$ a PL map from a p -conformal graph to a q -conformal graph, $E_q^p(\phi) = E_{q,\text{strip}}^p(\phi)$.

Proof sketch. This is closely related to Proposition A.11. For $p < q$, subdivide G_2 so that $\text{Fill}^p(\phi)$ is constant on each edge. Then for e an edge of G_2 , construct the strip structure (Γ_2, w_2, ℓ_2) compatible with G_2 so that

$$\ell_2(e) = \alpha_2(e) \cdot (\text{Fill}^p(\phi)(e))^{1/(q-p)}.$$

This determines ℓ_1 by the condition that ϕ be length-preserving, and w_1 and w_2 by the compatibility condition. It is elementary to check that ϕ is 1-filling with respect to these strip structures and then verify that $E_q^p(\phi) = E_{q,\text{strip}}^p(\phi)$. The case $p = q$ is easier, as you can choose ℓ_2 arbitrarily. \square

For the final variation on the definition of E_q^p , we allow more general spaces than graphs.

Definition A.24. For $1 \leq p \leq \infty$, a p -conformal space is (loosely) a tuple (X, ℓ, μ) of a space X , a length metric ℓ on X , and a measure μ on X , up to rescaling by

$$(X, \ell, \mu) \equiv_p (X, \lambda\ell, \lambda^p\mu)$$

for a suitable rescaling function $\lambda: X \rightarrow \mathbb{R}_{>0}$. Write $[(X, \ell, \mu)]_p$ for an equivalence class of \equiv_p .

There are analytic subtleties in Definition A.24 in, e.g., how to define the rescaling and exactly which metrics are allowed; we do not attempt to address them in this paper. But note that oriented conformal n -manifolds M^n give examples of n -conformal spaces: given a conformal class of (Riemannian) metrics on M , pick a base metric g in the conformal class, and set ℓ and μ to be distance with respect to g and the Lebesgue measure of g , respectively. Picking a different metric in the conformal class changes ℓ and μ by an n -conformal rescaling.

Suppose $X = \Gamma$ is a graph with a base metric and associated measure and

- ℓ is a piecewise-constant multiple of the base metric;
- μ is a piecewise-constant multiple of the base Lebesgue measure; and
- the rescaling functions λ are piecewise-constant.

Definition A.24 is then almost identical to Definition A.18, if we define the weight at a generic point $x \in \Gamma$ by

$$w(x) = \frac{\mu(\Delta x)}{\ell(\Delta x)}$$

where Δx is a small interval centered on x .

Definition A.25. For $1 \leq p \leq q \leq \infty$ and $\phi: X_1^p \rightarrow X_2^q$ a suitable map from a p -conformal space to a q -conformal space, with $X_i^p = [(X_i, \ell_i, \mu_i)]_p$, define

$$\begin{aligned} \text{Fill}_{\text{conf}}^p(\phi): Y^q &\rightarrow \mathbb{R}_{\geq 0} \\ \text{Fill}_{\text{conf}}^p(\phi) &:= \phi_*((\text{Lip}_{\ell_2}^{\ell_1}(\phi))^p \cdot \mu_1) / \mu_2 \\ E_{q,\text{conf}}^p(\phi) &:= (\|\text{Fill}_{\text{conf}}^p(\phi)\|_{q/(q-p), X_2})^{1/p} \end{aligned}$$

To take the definition of $E_{q,\text{conf}}^p$ step-by-step:

- $\text{Lip}_{\ell_2}^{\ell_1}(\phi): X_1 \rightarrow \mathbb{R}_+$ is the best local Lipschitz constant of ϕ .
- Next, $\phi_*((\text{Lip}_{\ell_2}^{\ell_1}(\phi))^p \cdot \mu_1)$ is the push-forward of the given measure on X_1 to a measure on X_2 .
- $\text{Fill}_{\text{conf}}^p(\phi) = \phi_*(\text{Lip}(\phi)^p \cdot \mu_1) / \mu_2$ is the Radon-Nikodym derivative of the two measures.
- Finally, $E_{q,\text{conf}}^p(\phi)$ is the $L^{q/(q-p)}$ -norm of $\text{Fill}_{\text{conf}}^p$.

We do not attempt to define which maps ϕ are “suitable”, but it should include cases where ϕ is Lipschitz and the Radon-Nikodym derivative exists, i.e., $\phi_*(\text{Lip}(\phi)^p \cdot \mu_1)$ is absolutely continuous with respect to μ_2 . For maps between graphs, PL maps are suitable.

For $q = \infty$ (so that X_2 is a length space), $E_{\infty, \text{conf}}^p$ can be rewritten

$$(A.26) \quad E_{\infty, \text{conf}}^p(\phi) = \left\| \text{Lip}_{\ell_2}^{\ell_1}(\phi) \right\|_{p, X_1}.$$

Note that we do not need the Radon-Nikodym derivative in (A.26).

The motivation for the specific exponents in Definition A.25 is that, up to an overall power, $E_{q, \text{conf}}^p$ is the unique expression constructed with this data and these operations that is invariant under both p -conformal rescaling on X_1^p and q -conformal rescaling on X_2^q .

Proposition A.27. *For $f: G^p \rightarrow H^q$ a PL map from a p -conformal graph to a q -conformal graph, $E_q^p(f) = E_{q, \text{conf}}^p(f)$.*

Proof. Follows from expanding the definitions. \square

Definitions A.24 and A.25 point to a considerably more general setting. Note that there are likely to be substantial new difficulties. As mentioned in Section 1.1, much prior attention has been devoted to proving the existence of harmonic maps between various types of spaces [EF01] (related to minimizing E_{∞}^2), and the general case is likely to be harder.

Warning A.28. E_{∞}^2 from Definition A.25 does not agree with standard definitions of Dirichlet energy. For instance, suppose X_1 is a Riemann surface Σ (with its natural 2-conformal structure), and X_2 is a Riemannian n -manifold M . Pick a base metric g on Σ in the given conformal class. Then, given a smooth map $f: \Sigma \rightarrow M$, we can consider the Jacobian $df_x \rightarrow T_x \Sigma \rightarrow TM$, and specifically its singular values $\lambda_1, \lambda_2: \Sigma \rightarrow \mathbb{R}_{\geq 0}$, the eigenvalues of $\sqrt{(df_x)^T(df_x)}$, chosen so that $\lambda_1(x) \geq \lambda_2(x)$. Thus df_x maps the unit circle in $T_x \Sigma$ to an ellipse whose major and minor axis have length $\lambda_1(x)$ and $\lambda_2(x)$. The local Lipschitz constant of f at x is $\lambda_1(x)$, so the formulas above give

$$(E_{\infty, \text{conf}}^2(f))^2 = \int_{\Sigma} \lambda_1(x)^2 \mu(x)$$

(where μ is Lebesgue measure on Σ) while the standard Dirichlet energy is

$$\text{Dir}(f) = \int_{\Sigma} (\lambda_1(x)^2 + \lambda_2(x)^2) \mu(x).$$

These energies are both conformally invariant, but are not the same. They do agree if the target space is a graph ($n = 1$), as in that case $\lambda_2(x) = 0$.

APPENDIX B. RELATION TO RESISTOR NETWORKS

The elastic graphs of this paper are closely related to the much better studied theory of *resistor networks*. Suppose we are given an elastic graph G with k marked vertices x_1, \dots, x_k , called *nodes*. Turn it into a network of resistors, where the elastic constants $\alpha(e)$ become resistances. If we attach external voltage sources at voltages V_1, \dots, V_k to the nodes, then the remainder of the circuit will reach an electrical equilibrium, which has several pieces of data:

- a voltage $V(v)$ for each vertex v of G (agreeing with V_1, \dots, V_k on the nodes);
- an internal current $I(\vec{e})$ flowing through each oriented edge \vec{e} of G (with $I(-\vec{e}) = -I(\vec{e})$);
- the total current I_1, \dots, I_k flowing out of the nodes; and
- the total energy E dissipated by the system per unit time.

At equilibrium, these are related by Kirchhoff's current laws.

- The current on an edge is related to the voltage difference. If \vec{e} has source s and target t , let $\Delta V(\vec{e}) = V(t) - V(s)$; then

$$I(\vec{e}) = \frac{\Delta V(\vec{e})}{\alpha(e)}.$$

- For each internal (unmarked) vertex v of G , the total current flowing in is 0

$$\sum_{\vec{e} \text{ incident to } v} I(\vec{e}) = 0,$$

while at the node x_i ,

$$I_i = \sum_{\vec{e} \text{ incident to } x_i} I(\vec{e}).$$

- The energy dissipated is

$$E = \sum_{e \in \text{Edge}(G)} \alpha(e) I(e)^2 = \sum_{e \in \text{Edge}(G)} \frac{(\Delta V(e))^2}{\alpha(e)}.$$

The energy dissipated is identical to Equation (1.7) for the Dirichlet energy of a map f , in the special case that the target of f is \mathbb{R} with k marked points at V_1, \dots, V_k .

For resistor networks, the equations for the internal voltages and currents are linear, so $V(v)$, $I(\vec{e})$, and I_i are linear functions of the V_i , while E is a quadratic function of the V_i . (By contrast, in the more general case considered in the bulk of this paper, the energy $\text{Dir}_{[f]}$ as a function of lengths is only piecewise-quadratic.) The *response matrix* of the resistor network is the matrix that gives the external currents I_i as a function of the V_i . This turns out to be a symmetric matrix that also determines E as a quadratic function of the V_i .

Much attention has been devoted to the question of when two resistor networks (with the same number of nodes) are *electrically equivalent*, in the sense that the response matrices are the same. Series and parallel reduction of resistors are examples of electrical equivalence. A more substantial example [Ken99] is the Y - Δ transform, that relates a 3-node network with the topology of a Y to one with the topology of a Δ :

$$(B.1) \quad \begin{array}{c} \otimes \\ | \\ r_1 \\ | \\ \otimes \quad r_2 \quad r_3 \quad \otimes \end{array} \quad \equiv_{\text{elec}} \quad \begin{array}{c} \otimes \\ \diagup \quad \diagdown \\ R_3 \quad R_2 \\ \diagdown \quad \diagup \\ \otimes \quad R_1 \quad \otimes \end{array}$$

where

$$\begin{aligned} r_1 &= \frac{R_2 R_3}{R_1 + R_2 + R_3} & R_1 &= \frac{r_2 r_3 + r_1 r_3 + r_1 r_2}{r_1} \\ r_2 &= \frac{R_1 R_3}{R_1 + R_2 + R_3} & R_2 &= \frac{r_2 r_3 + r_1 r_3 + r_1 r_2}{r_2} \\ r_3 &= \frac{R_1 R_2}{R_1 + R_2 + R_3} & R_3 &= \frac{r_2 r_3 + r_1 r_3 + r_1 r_2}{r_3} \end{aligned}$$

It turns out that the Y - Δ transform and series and parallel reduction are sufficient to relate any two electrically equivalent resistor networks if both are planar, with the nodes on the external face [CIM98, CdV94, CdVGV96].

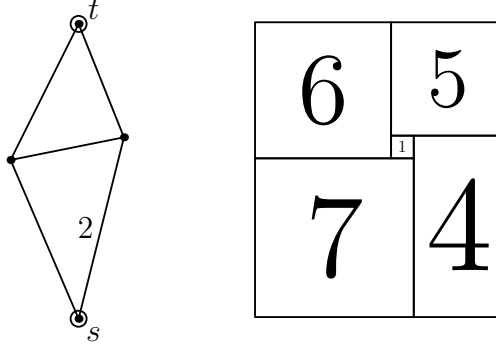
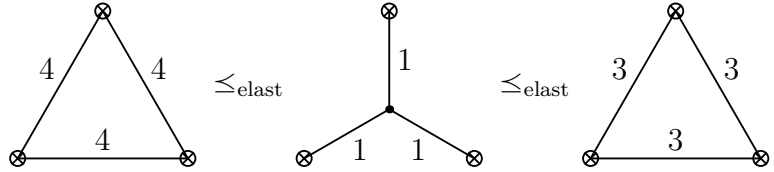


FIGURE 6. A simple electric network and an associated tiling by rectangles. All resistances on the graph are 1 except for one edge, which has resistance 2. On the rectangle tiling, we have shown the total current flowing through the edge, which in the picture is the width of the rectangles.

By contrast, elastic networks with targets more general than \mathbb{R} are almost never equivalent, so instead we can look for inequalities of the energy response function, as in Theorem 1. For instance, we have the following inequalities of energies:



where $G_1 \leq_{\text{elast}} G_2$ means that, for any homotopy class $[\phi_1]$ of maps from G_1 to a marked tree K and corresponding homotopy class $[\phi_2]: G_2 \rightarrow K$,

$$\text{Dir}[\phi_1] \leq \text{Dir}[\phi_2].$$

In fact, these inequalities hold more generally when K is any CAT(0) space with three marked points.

The second and third graphs are related by the Y – Δ relation of Equation (B.1), and so those two graphs have equal energies when K is \mathbb{R}^n for any n . (The case $n = 1$ is electrical equivalence, and for $n \geq 2$ the Dirichlet energy is the sum of the energies of the projections to the different coordinates.)

In general, a resistor network G with three nodes is always electrically equivalent to a tripod and Δ (related to each other by the Y – Δ transform). It turns out that, when we pass to elastic networks, the energy of G is always sandwiched between the energies of the tripod and Δ , at least when the target space is a tree [DG16].

We close by reminding the reader of the connection between electrical networks at equilibrium and rectangle tilings [BSST40]: loosely, if you assign each edge a rectangle of length equal to the voltage difference between the endpoints and width equal to the total current, then Kirchhoff’s laws say that the rectangles may be assembled into a single tiling, in which the aspect ratios are equal to the resistances. See Figure 6 for a simple example. In the more general setting of elastic graphs, the “weights” throughout this paper can also be reinterpreted as “widths”, giving similar tiling pictures. However, there are at least two additional complications.

- At vertices of the graph, the rectangle tiling necessarily has singularities, like zeroes of a quadratic differential.
- To get honest tilings, you need some additional structure, namely a ribbon structure on the graphs, and maps need to respect this ribbon structure. We need to check that energy minimizers respect the ribbon structure.

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